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QUANTITATIVE SYNTHESIS OF UNCERTAIN MULTIPLE INPUT-OUTPUT FEEDB--ETC(U)
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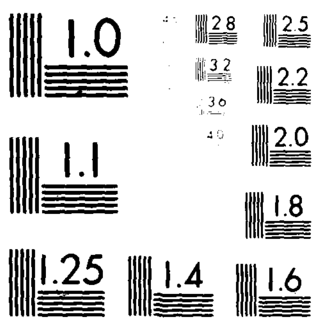

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1. INTRODUCTION

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There is great interest in multiple input-output (mio) feedback systems, for obvious reasons. A great deal of significant work (too numerous to list but Wonham and Morse 1972, MacFarlane 1973, Wang and Davison 1973, Rosenbrock 1974, Porter and D'Azzo 1978 are representative and include bibliographies) has been done, primarily in the realization and properties of the closed-loop input-output relations, under the constraint of a feedback structure around the known, fixed mio "plant." There has been notable work done with uncertain inputs, but again only with fixed, known plants. Of course, plant uncertainty is always implicit, if only because of the usual approximations required to obtain a linear time-invariant (ℓti) model.

In any case, there does not exist as yet any "quantitative synthesis" technique for the mio problem with significant plant uncertainty, even for the linear time-invariant case. By "quantitative synthesis" is meant that there are given quantitative bounds on the plant uncertainty, and quantitative tolerances on the acceptable closed-loop system response. The objective is to find compensation functions which guarantee that the performance tolerances are satisfied over the range of the plant uncertainty. In "quantitative design," one guarantees that the amount of feedback designed into the system is such as to obtain the desired tolerances, over the given uncertainty range. In other designs, the amount of feedback may be more or less than necessary-- it is a matter of chance. The practical experienced designer may find the

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QUANTITATIVE SYNTHESIS OF UNCERTAIN MULTIPLE INPUT-OUTPUT FEEDBACK SYSTEM

Isaac Horowitz*

ABSTRACT

There is given an n input, n output plant with a specified range of parameter uncertainty and specified tolerances on the n^2 system response to command functions and the n^2 response to disturbance functions. It is shown how Schauder's fixed point theorem may be used to generate a variety of synthesis techniques, for a large class of such plants. The design guarantees the specifications are satisfied over the range of parameter uncertainty. An attractive property is that design execution is that of successive single-loop designs, with no interaction between them and no iteration necessary. Stability over the range of parameter uncertainty is automatically included.

By an additional use of Schauder's theorem, these same synthesis techniques can be rigorously used for quantitative design in the same sense as above, for $n \times n$ uncertain nonlinear plants, even nonlinear time-varying plants, in response to a finite number of inputs.

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latter approach sufficient. However, a scientific theory of feedback should certainly include quantitative design techniques.

In this paper it is shown how Schauder's fixed point theorem can be used to generate a variety of precise quantitative mio synthesis techniques suitable for various problem classes. An outstanding feature of each synthesis procedure is that it consists of a succession of direct (no iterations necessary) single-loop design steps. Furthermore, by a second use of Schauder's theorem, the techniques are rigorously applicable to quantitative synthesis of nonlinear uncertain mio feedback systems. This paper concentrates on existence proofs but a 2 x 2 example is included.

1.1 Preliminary Statement of a Linear Time Invariant MIO Problem

In Fig. 1, $P = [p_{ij}(s)]$ is a n x n matrix of the plant transfer functions in the form of rational functions, each with an excess $e_{ij} > 0$ of poles over zeros, and with a bounded number of poles. The $p_{ij}(s)$ are functions of q physical parameters, with m an ordered real q-tuple sample of their values. $M = \{m\}$ is the class of all possible parameter combinations. The elements of the n x n lti compensation rational transfer function matrices $F = [f_{ij}(s)]$, $G = [g_{ij}(s)]$ are to be chosen practical (each with an excess of poles over zero). They must ensure that in response to command inputs the closed-loop transfer function matrix $T = [t_{uv}(s)]$ (of c = Tr) in Fig. 1 where c, r are the n x 1 matrices (vectors) of system outputs and inputs, respectively, satisfy conditions of the form

$$0 < A_{uv}(\omega) \le |t_{uv}(j\omega)| \le B_{uv}(\omega), \forall m \in M \tag{1}$$

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If the $t_{uv}(s)$ have no poles or zeros in the right half-plane (are stable and minimum-phase), then $t_{uv}(s)$ is completely determined by $|t_{uv}(j\omega)|$, so (1) suffices (Bode 1945). It has been shown (Horowitz 1976) that time-domain tolerances of the form

$$u_1^{\nu}(t) \le \frac{d^{\nu}\hat{c}(t)}{dt^{\nu}} \le u_2^{\nu}(t)$$

$\nu = 0, 1, \ldots, n_1$ any finite number, can be satisfied by means of tolerances like (1) on $|c(j\omega)|$, where $c(s) = \mathcal{L}\hat{c}(t)$. The writer finds it much more convenient to develop the synthesis theory in the frequency domain, and the above proves its sufficiency for time-domain synthesis.

This presentation concentrates on the command response problem, but the same ideas can be used to handle the quantitative disturbance response problem under plant uncertainty, as will be shown in Sec. 6. The constraints on the plant and the specifications are introduced as needed, in order to clarify the reasons for their need.

2. DERIVATION OF SYNTHESIS TECHNIQUE

In Fig. 1, there are available n^2 loop transfer functions in $L = [\ell_{ij}(s)] = PG$, and n^2 f_{ij} in F for satisfying the tolerances (1) on the n^2 t_{ij}. But in the expansion of $T = [t_{ij}(s)] = (I + L)^{-1}LF$, each $t_{ab}(s,m)$ $(m \in \mathcal{M})$ is a function of all the $\ell_{ij}(s,m)$ each uncertain, resulting in very complicated expressions for t_{ab} and making direct quantitative synthesis seemingly impossible--at least so far unsuccessful. The objective here is to convert each $t_{ab}(s,m)$ design problem into an equivalent single-loop problem with uncertainty. This is done for each t_{ab}, by lumping all the other inter-

acting t_{ij} variables into an 'equivalent disturbance', as follows.

In Fig. 1, $c = PG(Fr - c)$, so

$$(P^{-1} + G)c = GFr. \tag{2}$$

Hence, the following restriction on P:

(P1): $\Delta(s) \triangleq$ determinant $P(s) \not\equiv 0, \forall m \in M$.

Let $r_v \neq 0$ and $r_i \equiv 0$, $i \neq v$, so the resulting $c_j(s) = t_{jv}(s) r_v$. Let

$$P^{-1} = [P_{ij}(s)]. \tag{3}$$

The uth element of (2) is then

$$r_v(s) \sum_{i=1}^{n} (P_{ui} + g_{ui}) t_{iv} = \sum_i g_{ui} f_{iv}.$$

To simplify the presentation, we take $g_{ui} \equiv 0$ for $u \neq i$ (although in practice it may be useful not to do so). Then letting $r_v(s) = 1$, the last equation can be written as

$$t_{uv} = \frac{\frac{1}{P_{uu}} g_{uu} f_{uv} - \frac{d_{uv}}{P_{uu}}}{1 + \frac{g_{uu}}{P_{uu}}} \triangleq \tau_{uv} - \tau_{duv} d_{uv} \tag{4a}$$

$$d_{uv} = \sum_{i \neq u} P_{ui} t_{iv} \tag{4b}$$

This corresponds precisely to the single-loop problem of Fig. 2, with $P_{uve} = 1/P_{uu}$. Of course, the t_{iv} in d_{uv} of (4b) are not known but the bounds (1) on $|t_{iv}|$ are known, generating a set $D_{uv} = \{d_{uv}\}$. We define the extreme d_{uv}

$$|d_{uve}| = \sup_{M} \sum_{i \neq u} |P_{ui}||B_{iv}|, \; B_{iv} \text{ of } (1) \tag{5}$$

Suppose we can find $g_{uu}(s)$ and $f_{uv}(s)$, such that in the notation of (4,5)

$$0 < |\tau_{uv}| \pm |\tau_{duv}||d_{uve}| \varepsilon \, [A_{uv}, B_{uv}], \forall \, m \varepsilon M \qquad (6)$$

Then the magnitude of the right side of (4a) $\varepsilon[A_{uv}, B_{uv}]$ for all $m \varepsilon M$ and for all possible combinations of t_{iv} $(i \neq u)$ which satisfy (1). Suppose this is so $\forall$ u,v combinations, and the other Schauder conditions of Sec. 2.1 are satisfied. Then Schauder's fixed point theorem can be used to prove that these same n g_{uu} and n^2 f_{iv} are a solution to the synthesis problem (1).

2.1 Application of Schauder's Fixed Point Theorem

This theorem states that a continuous mapping of a convex, compact set of a Banach space into itself, has a fixed point (Kantorovich and Akilov 1964). We define the Banach space to be the n^2 $C[0,\infty]$ product space denoted here by $C(n^2)$, with norm = Σ individual sup norms. $C[0,\infty]$ is the Banach space of real continuous functions $f(\omega)$, $\omega \varepsilon [0,\infty]$ with $||f|| = \sup_{\omega} |f(\omega)|$. The convex compact set in each of the n^2 $C[0,\infty]$ is taken as the acceptable set of $|t_{uv}(j\omega)|$ satisfying (1), denoted by $\{h_e(\omega)\} = H_{uv}$. Additional constraints have to be assigned to the h_e (ω) in order that each H_{uv} set is compact and convex in $C[0,\infty]$. These constraints have been justified in detail in (Horowitz 1975) and are therefore only summarized here. If each set is convex and compact in $C[0,\infty]$, their n^2 product set denoted by $H(n^2)$ is convex and compact in $C(n^2)$.

Constraints on $H_{uv} = \{h(\omega)\}_{uv}$

1. $\exists$ continuous functions $A_{uv}(\omega)$, $B_{uv}(\omega)$ with properties of (1) as bounds on $h(\omega)$
2. $h'(\omega)$ is uniformly bounded: $\exists \, K, \ni \; |h'(\omega)| < K, \forall \, h,\omega$

3. $h(\omega) \to o$ as $\omega \to \infty$ in the form k/ω^e, e a fixed finite number $\geqslant 3$ to allow at least one excess of pole over zeros for the elements of F,G,P in Fig. 1.

These constraints guarantee (Horowitz 1975) that $h(\omega)$ can be taken as the magnitude of a function $\hat{h}(s)_{s=j\omega}$ which has no zeros or poles in the interior of the right half-plane or on the $j\omega$ axis. Arg $\hat{h}(j\omega)$ is obtained from $h(\omega)$ by anyone of a number of Bode integrals (Bode 1945).

An element of $H(n^2)$ consists of n^2 positive functions on $[0,\infty]$, $h_{ik}(\omega)$. Using any appropriate Bode integral, find the associated phase function denoted here by $\arg[h_{ik}(\omega)]$, giving the minimum-phase stable function $\hat{h}_{ik}(s)$, $\hat{h}_{ik}(j\omega) = h_{ik}(\omega) + j \arg[h_{ik}(\omega)]$. For future use, denote this sequence of operations whereby $h(\omega)$ is transformed into $\hat{h}(j\omega)$, as the "Bode transformation" $B(h(\omega))$. Define Φ on $H(n^2)$ by

$$\Phi = (\psi_{11}, \psi_{12}, \ldots, \psi_{nn}): \; H(n^2) \to H(n^2), \; \psi_{uv}(h_{11}, h_{12}, \ldots, h_{nn})$$

$$= \left| \frac{g_{uu} f_{uv} - \sum_{i \neq u} P_{ui} B(h_{iv}(\omega))}{P_{uu}\left(1 + \frac{g_{uu}}{P_{uu}}\right)} \right| \tag{7}$$

using for P_{ui}, P_{uu} any specific fixed $m \epsilon M$. (Note the similarity of (7) to (4a,b)). In Appendix 2, it is shown that g_{uu}, f_{uv} can be found such that Φ maps $H(n^2)$ into itself. It is also necessary to prove Φ is continuous, as follows.

<u>Φ is a continuous mapping</u>

Φ is continuous if each of its n^2 components is continuous. The first step in each mapping is $B(h_{iv}(\omega)) = \hat{h}_{iv}(j\omega)$. In (Horowitz 1975, Sec. III) it is proven that the step $h_{iv}(\omega) \to \arg h_{iv}(\omega) \stackrel{\Delta}{=} \theta_{iv}(\omega)$ is continuous in the $C[0,\infty)$ norm. Hence,

the mappings $h_{iv}(\omega) \to h_{iv}(\omega)\cos\theta_{iv}(\omega) \triangleq \mathcal{R}_{iv}(\omega)$, $h_{iv}(\omega) \to h_{iv}(\omega)\sin\theta_{iv}(\omega) \triangleq X_{iv}(\omega)$ are continuous. The denominator of (7) is a constant on $H(n^2)$, and so are $g_{uu}\, f_{uv}$ and the P_{ui} in the numerator. Thus, the numerator has the form

$$\text{Num.} = |K_a + jK_b - \sum_i (C_i + jD_i)(\mathcal{R}_i(\omega) + jX_i(\omega)), \; j = \sqrt{-1},$$

all other terms real and only the $\mathcal{R}_i$, X_i mappings on $H(n^2)$. Infintesimal changes in $\mathcal{R}_i$, X_i clearly result in similar change in Num., so Num. is continuous on $H(n^2)$ and so is each ψ_{uv} of (7) and hence Φ. The conditions in Schauder's theorem are satisfied, so Φ has a fixed point.

This means $\exists$ a set of $h_{ij}(\omega)$ denoted by $h^*_{ij}(\omega)$, $\ni$

$$h^*_{uv}(\omega) = \left| \frac{g_{uu} f_{uv} - \sum_{i \neq u} P_{ui} \hat{h}^*_{iv}(j\omega)}{P_{uu}\left(1 + \frac{g_{uu}}{P_{uu}}\right)} \right| \tag{8}$$

$u, v = 1, \ldots, n$, where $\hat{h}^*_{iv}(j\omega) = B(h^*_{iv}(\omega))$.

We would now like to deduce from (8), that

$$B(h^*_{uv}(\omega)) = h^*_{uv}(j\omega) = \frac{g_{uu} f_{uv} - \sum_{i \neq u} P_{ui} \hat{h}^*_{uv}(j\omega)}{P_{uu}\left(1 + \frac{g_{uu}}{P_{uu}}\right)} \qquad (9. ?)$$

For, if (9) is true, then by letting $\hat{h}^*_{uv}(j\omega) = t_{uv}(j\omega)$, we have recovered (4) and the n^2 $\hat{h}^*_{uv}(j\omega)$ are a solution to the mio problem for that specific $m \in M$. The solution is unique if every building block in the mio system has a unique output for any given input, which is a very reasonable condition. This makes

it unnecessary to prove that there are no transitions from (8) to an expression similar to (9) but with right half plane poles and/or zeros. Since m is any element of M, this is true for all $m \varepsilon M$ (of course with a different set of $\hat{h}^*_{uv}$ for each m).

The step from (8) to (9) is a crucial one and must be justified with great care. Given an analytic function $\phi(s)$, there is an infinitude of $\psi(s)$ such that $|\phi(j\omega)| = |\psi(j\omega)|$, $\omega \varepsilon [0,\infty]$, e.g.

$$\psi(s) = \phi(s) \frac{(1 - \tau_1 s)}{(1 + \tau_1 s)} \frac{(1 + \tau_2 s)}{(1 - \tau_2 s)}$$

But $\phi(s) \not\equiv \psi(s)$ even though $|\phi(j\omega)| \equiv |\psi(j\omega)|$. But suppose we know from other sources that $\phi_1(s)$ has no right half plane zeros or poles, then given $|\phi_1(j\omega)| \equiv M(\omega)$ a magnitude function which is Bode transformable, we can conclude that $\phi_1(j\omega) \equiv B(M(\omega)) = \hat{M}(j\omega)$. Hence, to justify (9) we must prove that the expression inside the vertical bars in (8) has no right half-plane zeros or poles. The pole part is easy, because $1 + g_{uu}/P_{uu}$ is obviously designed to have no right half-plane zeros; certainly g_{uu}, f_{uv} won't be assigned any such poles; $\hat{h}_{iv}(s)$ doesn't have any by definition, and P_{ui} is not allowed any such poles--see Sec. 3.1. To prove the zero part, note that from (6) and Rouche's theorem, the number of zeros of the right side of (9) in the right half-plane, equals such number of

$$\frac{g_{uu} f_{uv}}{P_{uu}\left(1 + \frac{g_{uu}}{P_{uu}}\right)},$$

which is easily made zero in the single-loop synthesis steps (if P_{uu} has no right half-plane poles, a condition necessary for other reasons--see Sec. 3.1). Thus, the expression inside the bars in (8) has no right half-plane poles or zeros, justifying (9). This is a very valuable result. The problem of stabilizing a highly uncertain n x n mio system is automatically disposed of in the synthesis procedure, which is furthermore one of designing n single-loop transmission functions.

It is worth noting that even if the above proof was not available, it would not be disastrous for this synthesis theory. It would only be necessary to guarantee that at <u>one</u> $m \epsilon \mathcal{M}$, the system is stable and minimum-phase. For then, this would be so $\forall\, m \epsilon \mathcal{M}$, because by the continuity of the poles (and zeros) with respect to the parameters, the right side of (8) would have to be infinite (zero) at some ω, in order that for some $m \epsilon \mathcal{M}$ the system should be unstable (or have a right half-plane zero). However, the synthesis procedure by definition precludes this. And it is a relatively easy matter to guarantee the desired conditions at one $m \epsilon \mathcal{M}$.

3. CONSTRAINTS ON MIO PLANT

The above results hinge on our ability (a) to find g_{uu} and f_{uv} to satisfy (6) $\forall\, \omega$, all u,v pairs and all $m \epsilon \mathcal{M}$ (b) that each equivalent single-loop design is stable and minimum-phase $\forall\, m \epsilon \mathcal{M}$. These lead to constraints on the mio plant, obtained by applying single-loop design theory to achieve (a,b). Appendix 1 gives an existence theorem for single-loop design. The first part of the design (see Appendix A3) gives bounds on the nominal loop transmission which is g_{uu}/P_{uuo} of (4a), where P_{uuo} is the 'nominal' associated with a nominal $m_o \epsilon \mathcal{M}$.

These bounds must be satisfied in order that a specific system transfer function t_{uv} satisfy (1). Here g_{uu}/P_{uuo} is used for n t_{uv} ($v = 1,\dots,n$) functions. It is proven in A3, that a g_{uu}/P_{uuo} can be found which satisfies the conditions for all n t_{uv} functions.

For example, consider t_{u1} at $\omega = \omega_1$ and suppose $A_{u1}(\omega_1) = .9$, $B_{u1}(\omega_1) = 1.1$ in (1). We could split this range [.9, 1.1] into say [.95, 1.05] for τ_{u1} and .05 for $\tau_{du1}d_{u1}$ in (4), using d_{u1e} of (5) for d_{u1}. The technique in A3 or better (Horowitz and Sidi 1972), is then used to find a bound on $g_{uu}(j\omega_1)$. Here, we note a tough constraint. Sooner or later in ω, $|g_{uu}(j\omega)|$ must become very small with $1 + g_{uu}/P_{uu} \to 1$ and then in (4a)

$$t_{uv} \to \frac{g_{uu}f_{uv} - d_{uv}}{\mathcal{P}_{uu}} \tag{10}$$

and in (7), $\psi_{uv} \to$ the numerator of its right side divided by $\mathcal{P}_{uu}$. Now (4a, 5, 6) in general require that

$$|t_{uv}|_{max} > 2|\tau_{duv}d_{uve}| \tag{11}$$

But $|t_{uv}|_{max} = B_{uv}$ and at high frequencies

$$|\tau_{duv}d_{uve}| \to \frac{\sup_{M} \sum_{i \neq u} |\mathcal{P}_{ui}||B_{iv}|}{|\mathcal{P}_{uu}|} .$$

To see what this leads to take, for example, n = 2 so that the above applied to v = 1, u = 1,2 gives

$$B_{11} > \frac{2|\mathcal{P}_{12}|B_{21}}{|\mathcal{P}_{11}|}, \qquad B_{21} > \frac{2|\mathcal{P}_{21}|B_{11}}{|\mathcal{P}_{22}|},$$

requiring

$$1 > \frac{4|P_{12}P_{21}|}{|P_{11}P_{22}|} \qquad \text{as } \omega \to \infty \tag{12}$$

Thus, a constraint on $\mathcal{P}$ is

$$(P_{2a}): \quad \exists\, \omega_h, \ni \text{ for } \omega > \omega_h, \ |P_{11}P_{22}| > 4|P_{12}P_{21}| \forall m \epsilon M. \tag{13}$$

It is known that as $s \to \infty$,

$$P_{ij} \to \frac{k_{ij}}{s^{e_{ij}}},$$

so the above becomes

$$\frac{|k_{11}k_{22}|}{\omega^{e_{11}+e_{22}}} > \frac{4|k_{12}k_{21}|}{\omega^{e_{12}+e_{21}}}.$$

If the uncertainties in the k_{ij} are independent and $e_{11} + e_{22} = e_{12} + e_{21}$, this becomes

$$k_{11min}k_{22min} > 4k_{12max}k_{21max}. \tag{14}$$

There is an important problem class for which the inequality is less harsh. This is the "basically noninteracting" class, where one ideally desires $t_{ij} \equiv 0$ for $i \neq j$, but because of uncertainty accepts $A_{ij} = 0$, $|t_{ij}| \le B_{ij}$ for $i \neq j$, in (1). Also, one doesn't care if $t_{ij} (i \neq j)$ is nonminimum-phase. Condition (6) then applies only to $u = v$. The f_{uv} $(u \neq v)$ are set equal to zero and (13) becomes

$$\exists\, \omega_h, \ni, \ |P_{11}P_{22}| > 2|P_{12}P_{21}| \ m \epsilon M, \ \omega > \omega_h. \tag{15}$$

It is desirable to ease inequality (13) in the general case. Note that (6) can be satisfied over any finite ω range by making $|1 + g_{uu}/\mathcal{P}_{uu}|$ large enough. Thus, as previously indicated, one can split the $[A_{uv}, B_{uv}]$ tolerance so that $|\tau_{uv}| > |\tau_{duv}||d_{uve}|, \forall\, m \varepsilon M$, e.g. assign $|\tau_{uv}| \varepsilon [E - \varepsilon, E + \varepsilon]$ with $E = (A_{uv} + B_{uv})/2$, $2\varepsilon < B_{uv} - A_{uv}$ and the balance $(B_{uv} - A_{uv} - 2\varepsilon)/2$ is assigned to $\tau_{duv} d_{uv}$ of (4a). But $|1 + g_{uu}/\mathcal{P}_{uu}|$ must then be made large enough to satisfy the resulting requirements, and it can for any finite ω range. The trouble is that g_{uu} must be allowed to $\to$ zero as $\omega \to \infty$, leading to (13), etc., if we insist on (6). We could ignore (6) at large ω, say for $\omega > \omega_H$, with ω_H as large as desired but finite, letting $|\tau_{uv}| << |\tau_{duv}||d_{uve}|$ for $\omega > \omega_H$. Then for $\omega > \omega_H$, (11) is replaced by the weaker

$$|t_{uv}|\max_{M} > |\tau_{duv} d_{uve}| \tag{16}$$

and for n = 2, (13) is then replaced by

$$(P_{2b}):\ \exists\ \omega_h, \ni \text{ for } \omega > \omega_h,\ |P_{11}P_{22}| > |P_{12}P_{21}|, \forall\, m \varepsilon M \tag{17a}$$

An important question is whether (17a) is an inherent basic constraint in the presence of uncertainty, no matter what design technique is used, or is due only to this specific design technique. The methods suggested in (Rosenbrock 1974, Owens 1978) to achieve diagonal dominance, may be helpful in satisfying (17a), but they would have to be extended to uncertain plants. Note that in Rosenbrock 1974, Owens 1978), diagonal dominance is desired $\forall\ \omega \ \varepsilon\ [0,\infty)$, whereas in (P_{2b}) it is required only for $\omega > \omega_H$.

For the analog of (17a) at n = 3, it is found that diagonal row dominance of P^{-1} for $\omega > \omega_H$, is a sufficient condition. The necessary condition can be written as

$\exists\ \omega_H,\ \ni$ for $\omega > \omega_H$ $|P_{ii}P_{jj}| > |P_{ij}P_{ji}|$ and

$$|P_{11}P_{33}| > (|P_{12}P_{23}| + |P_{13}P_{22}|)(|P_{22}P_{31}| + |P_{21}P_{32}|) \tag{17b}$$

which can be written as,

$$|P_{11}P_{22}P_{33}| > |P_{11}P_{23}P_{32}| + |P_{12}P_{21}P_{33}| + |P_{12}P_{23}P_{31}|$$

$$+ |P_{13}P_{22}P_{31}| + |P_{13}P_{21}P_{32}| \quad \text{for } \omega > \omega_H\ . \tag{17c}$$

The latter has the following intepretation. Array the matrix P^{-1} in the usual manner, but twice -one under the other as in Fig. 3a. Then the terms on the right side of (17c) consist of the products of the entries crossed by the dashed lines.

However, if ω_H is so used, it is no longer possible to use Rouche's theorem and thereby prove each t_{ij} is minimum-phase. But we can still design so that the nominal t_{ij} are minimum-phase and we know from (6) that $t_{ij}(j\omega) \neq 0$ for $\omega\epsilon[0,\omega_H]$. Therefore, from the continuity of the zeros of t_{ij} with respect to the parameters of the system, if t_{ij} has any right half-plane zeros, they must enter the right half-plane as shown in Fig. 3b. It is unlikely that such a zero which must migrate all the way up to $j\omega_H$, should move back into the significant control bandwith region A. The point is that if right half-plane zeros are "far-off", they have little effect and the system is "dominantly" minimum-phase.

Rouche's theorem can still be used if we can guarantee that (6) is satisfied for a semicircle consisting of the segment $[-j\omega_H, j\omega_H]$ and the right half-plane half-circumference of the circle of radius ω_H, centered at the origin. Then, there are definitely no right half-plane zeros of t_{ij} in this half-circle, and the system is "dominantly" minimum-phase This is quite practical in the design technique of (Horowitz and Sidi 1972), discussed in A3.

3.1 Modification of mapping Φ

Note that for the "dominantly minimum-phase" and the "basically noninteracting" cases, the application of Schauder's theorem in (2.1), Eqs. (7-9), etc., needs modification, because nonminimum-phase $t_{uv}(j\omega)$ cannot be uniquely derived from $|t_{uv}(j\omega)|$. Redefine $h \in H_{uv}$ of 2.1 to consist of an ordered pair: $h(\omega)$ as before and $q(\omega)$, the imaginary part of $\hat{h}_{uv}(j\omega)$ with $h = |\hat{h}_{uv}(j\omega)|$; $h \in H_{uv}$ the same as before but $q(\omega) \in C\,[0,\infty)$ with $0 \leq |q(\omega)| \leq h(\omega)$. Constraints 2,3 in 2.1 on $h(\omega)$ also apply to $q(\omega)$. Let $(HQ)_{uv} \subset C^2\,[0,\infty)$ denote the set $\{(h(\omega), q(\omega))\}$ with $||(h,q)|| = ||h|| + ||q||$. Obviously, $(HQ)_{uv}$ is compact and convex in $C^2\,[0,\infty)$. The extension to the n^2 product set is straightforward.

The mappings ψ_{uv} in (7) are redefined. Each ψ_{uv} is a pair of mappings, one the absolute value as before, the second the imaginary part with the absolute bars on the right removed. On the right side of (7), $B(h_{iv}(\omega))$ is replaced by $r_{iv}(\omega) + jq_{iv}(\omega)$, with $h_{iv}^2 = r_{iv}^2 + q_{iv}^2$, $(h_{iv}, q_{iv}) \in (HQ)_{iv}$. It is necessary to prove that Φ maps each element of $(HQ)_{uv}$ into itself.

The proof follows immediately from that for the minimum-phase case -- this is obvious from (6), the definition of d_{uve} in (5), and Appendices 1,2. The proof that Φ is continuous is straightforward. Accordingly, the Schauder conditions are satisfied and there exists a fixed point which satisfies the specifications. Such specifications, by themselves, would not be good ones because they permit highly nonminimum-phase $t_{uv}(s)$. However, they are satisfactory if it is known from other sources that t_{uv} is "dominantly minimum-phase".

3.2 Additional Constraints on P

Constraints A1(1)-(3) in the Appendix, must be applied to the $1|P_{uu}$, since in Fig. (2) $p_{uve} = \frac{1}{P_{uu}} = p$ of Appendix. A1.1 requires that there be no change in the excess of poles over zeros of $\frac{1}{P_{uu}} = \frac{\Delta}{\Delta_{uu}}$ where Δ = det. P and Δ_{uu} its uuth minor, as m ranges over $\mathcal{M}$. Also, that for at least one $m \epsilon \mathcal{M}$, denoted by m_{uo}, P_{uu} has all its poles and zeros in the interior of the left half-plane. The m_{uo} can be different for each u.

A1.2 requires that $1/P_{uu}$ is minimum-phase $\forall\ m \epsilon \mathcal{M}$, and its zeros do not get arbitrarily close to the $j\omega$ axis. Since $1/P_{uu} = \Delta/\Delta_{uu}$, this means Δ must have no right half-plane zeros. Hence the P_{ij} in general have no right half-plane poles. (For those who wish it, P is restricted to be controllable and observable $\forall\ m \epsilon \mathcal{M}$, but these concepts are unnecessary if P is properly formulated in terms of physical uncertain parameters (Horowitz and Shaked 1975)). Since the p_{ij} in $P = [p_{ij}]$ are finite rational functions, the latter part of A1.2 is automatically satisfied.

A1.3 for n = 2 is the same as (17), which shows that (17) is a fundamental condition for linear time-invariant design, not an "extra" condition due to our design technique, at least for n = 2. However, (13) is an "extra" condition. Note, the extension of single-loop design to disappearing poles and zeros in A6 may perhaps permit disappearing poles and zeros in the mio plant functions.

4. OTHER DESIGN EQUATIONS

The previous design equations constitute only one of many design techniques derivable from Schauder's fixed point theorem. Only two more will be briefly mentioned here.

Both are based on the use of a nominal diagonal loop transmission matrix. The design obligations on the loop transmission elements are then independent of the way the plant input and output terminals are numbered. If G is made diagonal, such numbering is important and after one arbitrarily numbers the plant input terminals, he should try to number the outputs such that the main effect of input i is on output i. Manipulation of (2) somewhat differently from Sec. 2, gives

$$t_{11} = \frac{f_{11}\ell_{11}/\delta_{11} + \sum_{i\neq 1} v_{1i}t_{i1}/\delta_{11}}{1 + \ell_{11}/\delta_{11}} \tag{18}$$

$$t_{21} = \frac{f_{21}\ell_{22}/\delta_{22} + \sum_{i\neq 2} v_{2i}t_{i1}/\delta_{22}}{1 + \ell_{22}/\delta_{22}}, \text{ etc.}$$

where $V = [v_{ij}] = I-\mathcal{P}_0(\mathcal{P})^{-1}$, P_o is the 'nominal' plant matrix and therefore fixed, P is the general uncertain plant matrix, $\delta_{ii} = 1 - v_{ii}$. The ℓ_{ii} are the nominal elements of the loop transmission matrix L. Eqs. (18) lend themselves to single-loop design and use of Schauder's theorem, pricely as did (4).

Another interesting set of design equations is obtained by designing to control the changes in t_{ij}, rather than t_{ij} directly. Let $T_o = [t_{ijo}]$ be the 'nominal' system transfer matrix and $T = [t_{ij}]$ the actual which is uncertain, $\Delta T = [\Delta t_{ij}] = T - T_o$. Then it can be shown that

$$\Delta T = (I+L)^{-1}\, VT, \quad V = I-\mathcal{P}_o\mathcal{P}^{-1} \tag{19}$$

where $\mathcal{P}_o$, $\mathcal{P}$ are likewise the 'nominal' and uncertain plant transfer matrices, and $L = \mathcal{P}_oG = [\ell_{ij}]$ is the <u>nominal</u> loop transmission matrix. If L is taken diagonal, the result is (n = 2 for simplicity)

$$\Delta t_{11} = \frac{v_{11}t_{11} + v_{12}t_{21}}{1 + \ell_{11}}, \quad \Delta t_{12} = \frac{v_{11}t_{12} + v_{12}t_{22}}{1 + \ell_{11}} \tag{20}$$

and similar obvious ones for Δt_{21}, Δt_{22}.

The design problem is now completely one of disturbance attenuation, with the disturbances $d_{11} = v_{11}t_{11} + v_{12}t_{21}$, etc., whose range is known. Schauder's theorem is applicable in the same manner as before. Note that V represents the 'normalized' plant variation matrix. Eqs. (20) appear to be much simpler to use for design (once

the Δt_{ij} tolerances are formulated) than (4), and their use needs to be intensively researched. However, both for (18) and (20) the constraints considered in 3., leading to (11-15) must be found, and these may possibly be tougher than before. Also, both a nominal $\underline{P}$ and T must be chosen, which is not good, because the optimum pairing is not apriori known. However, the analogs of (14,17) may be more lenient.

4.1 Bandwidth Minimization

An important criterion for comparison of design techniques is their "cost of feedback," which we take as the bandwidths of the loop transmission functions--because they determine the system sensitivity to sensor noise. Obviously, quantitative synthesis techniques must first be invented before one can turn to their optimization (for without such quantitative techniques comparison is possible at best, by analysis after a specific numerical design has been made). This approach via Schauder's theorem promises to generate a variety of such techniques, and the next step will be optimization.

5. DESIGN EXAMPLE

The 2 x 2 plant elements are $p_{ij} = k_{ij}/(1+sA_{ij})$ with correlated uncertainties, giving a total of 9 parameter sets in Table 1. The design was performed to handle the convex combination generated by these 9 sets (Figure 6).

TABLE I

No.	k_{11}	k_{22}	k_{12}	k_{21}	A_{11}	A_{22}	A_{12}	A_{21}
1.	1	2	.5	1	1.	2	2	3
2.	1	2	.5	1	.5	1	1	2
3.	1	2	.5	1	.2	.4	.5	1
4.	4	5	1	2	1.	2	2	3
5.	4	5	1	2	.5	1	1	2
6.	4	5	1	2	.2	.4	.5	1
7.	10	8	2	4	1.	2	2	2
8.	10	8	2	4	.5	1	1	2
9.	10	8	2	4	.2	.4	.5	1

A "basically noninteracting" system is desired, with the off-diagonal transmissions specified in the ω-domain $|t_{12}(j\omega)|$, $|t_{21}(j\omega)|<0.1\ \forall\omega$. The diagonal t_{11}, t_{22} bounds are identical and were originally in the time-domain in the form of tolerances on the unit step response shown in Fig. 4a, b (which also shows the design results for those of the 9 cases which were reasonably distinguishable). These time-domain bounds were translated into the "equivalent" bounds on $|t_{ii}(j\omega)|$ shown in Fig. 5 (Horowitz and Sidi 1972, Krishnan and Cruickshank 1977).

Familiarity with quantitative single-loop design is assumed here. One can do a problem of this complexity by hand. The sets $\{p_{iie}(j\omega)\}$, called the plant templates, are obtained on the Nichols chart. Some of these templates of $\underline{P}_{11}^{-1} = \frac{\Delta}{P_{22}}$, $\underline{P}_{22}^{-1} = \frac{\Delta}{P_{11}}$

are shown in Fig. 6 at various ω values. The larger the template, the greater uncertainty at that ω value. The tolerances on t_{uu} of (4a) and Fig. 5 were divided between τ_{uu} and $\tau_{duu}d_{uu}$ as discussed in Sec. 2. Each of these, in conjunction with the templates, leads to bounds on the nominal loop transmission $\ell_{u\omega o} = \frac{g_{uu}}{P_{uuo}}$. Some of these boulds on ℓ_{iio}, due to τ_{11}, are shown as solid lines in Fig. 7, i.e., it is necessary for ℓ_{11o} to lie above the indicated boundary. The tolerances on $\tau_{duu}d_{uu}$ lead to the dashed line bounds on ℓ_{11o}. No attempt was made to optimize the division of the tolerances between τ_{11} and $\tau_{d11}d_{11}$. The composite bound on ℓ_{11o} must satisfy both. The $\ell_{11o}(j\omega)$ chosen is also shown in Fig. 7. There was no attempt made to optimize the ℓ_{iio}; the design was made by hand quickly, so the $\ell_{iio}(j\omega)$ are larger than need be, with the tolerances therefore satisfied better than necessary--as seen in Figs. 4a, b. Optimal $\ell_{iio}(j\omega)$ would lie on their boundaries at each ω, so in this example there is considerable overdesign.

Here we took

$$\ell_{11o} = \frac{\Delta_o}{P_{22o}} g_{11} = \frac{10}{s} \frac{(1+.007s)}{(1+.025s)\left[1+\frac{s}{400} + \frac{s^2}{(400)^2}\right]}$$

with

$$\frac{\Delta_o}{P_{22o}} = \frac{.75\ (1+3.66s)}{(1+s)(1+3s)} \ ;$$

$$\ell_{22o} = \frac{\Delta_o}{P_{11o}} g_{22} = \frac{9}{s} \ \frac{(1+.02s)}{(1+.1s)\left[1+\frac{s}{150}+\frac{s^2}{(150)^2}\right]}$$

with

$$\frac{\Delta_o}{P_{11o}} = \frac{1.5\ (1+3.66s)}{(1+3s)(1+2s)} \ .$$

The requirements on f_{11}, f_{22} ($f_{12} = f_{21} = g_{12} = g_{21} = 0$ here) were found using single-loop design technique [15] as briefly explained here in A4, and

$$f_{11} = \frac{1}{1+.5s}\ , \quad f_{22} = \frac{1}{1+.33s}$$

were found satisfactory. The system was simulated on the digital computer with the results shown in Figs. 4a, b. The t_{12}, t_{21} tolerances were easily satisfied by the design.

While this is not a very challenging example of the design technique, nevertheless the uncertainty is very large and one should consider how quick, simple and straightforward was the design procedure, and also consider what alternatives are offered in the mio literature. There are no other techniques available for systematic design to specifications in the presence of significant uncertainty, which guarantee design convergence and attainment of design tolerances.

Whatever present popular technique is used, it would be necessary to cut and try and endeavor to understand the relations between the cutting and the results as one continued to cut and try, because these techniques have no provision for significant uncertainty. In the above design, one sweep was known to be sufficient because the plant and the design tolerances (ω-domain) satisfied constraints, P1 etc.

5. EXTENSION TO NONLINEAR UNCERTAIN MIO PLANTS

Once there is a quantitative design technique for linear time invariant mio uncertain plants, it appears at least conceptually possible to extend it to a significant class of nonlinear, even *nonlinear time-varying, uncertain mio plants. The procedure is a* generalization of that used (based also on Schauder's theorem) in (Horowitz 1976) for single loop uncertain nonlinear systems. The key feature is the replacement of the nonlinear plant matrix set (a set because of the uncertainty), by a linear time invariant plant set which is precisely equivalent to the original nonlinear set, with respect to the acceptable system output set. The procedure is briefly presented for the case where one wants the system with nonlinear uncertain plant to behave like a linear time-invariant system for a specified class of command input sets.

It is essential that the command input sets represent a good sampling of how the system will actually be used. For example, suppose $n = 3$ and in actual use r_1, r_2 always exist simultaneously (with $r_3 = 0$), and r_3 appears by itself (with $r_1 = r_2 = 0$). Say there are ten typical $r_1(t)$ inputs and for each typical $r_1(t)$ there are five typical $r_2(t)$. This makes a subtotal of 50 input sets, to which is added the number of typical $r_3(t)$ say 10, giving a class $R = \{\bar{r}\}$ of 60 sets, of which 50 have the form $\bar{r} = (r_1, r_2, 0)$ and $\bar{r} = (0, 0, r_3)$ for the balance. Choose $\bar{r}_1 \epsilon R$. The family of acceptable outputs for this input, is known from the tolerances on t_{ij}, giving for that <u>one</u> input vector a family $\mathcal{H} = \{\bar{h}\}$, $\bar{h} = (h_1, h_2, h_3)$. The mio plant is represented by a family (because of parameter uncertainty) $\mathcal{W}$ of nonlinear differential mappings

$$\mathcal{W} = \{W\}, \; W = (w_1, w_2, w_3); \; c_1 = w_1(x_2, x_2, x_3, m), \; . \; . \; . \; , \; c_3 =$$

$w_3(x_1, x_2, x_3, m)$, where the x_i are the plant inputs, c_i the plant outputs, and m is the plant parameter vector $m \epsilon M$.

Take a sample acceptable outputtriple $\bar{h} = (h_1, h_2, h_3)$ and find the corresponding plant inputs at some specific $m \epsilon M$ (or in other words, pick a $W \epsilon \mathcal{W}$) and let $c_j = h_j$ and solve the nonlinear equations backwards, giving the input set (x_1, x_2, x_3). Take the Laplace transforms $\hat{x}_i(s)$ of x_i, $\hat{h}_j(s)$ of h_j giving the vectors $\hat{x}[s] = (\hat{x}_1(s), \hat{x}_2(s), \hat{x}_3(s))$ $\hat{h}[s] = (\hat{h}_1(s), . . . , \hat{h}_3(s))$. Repeat for other $\bar{h}$ samples in the acceptable output set H, giving two paired families of $\hat{x}[s]$, $\hat{h}_j[s]$.

Select any combination of three $\hat{x}[s]$, forming a 3 x 3 matrix $\hat{X}$ and corresponding paired combination of three $\hat{h}[s]$, forming the matrix $\hat{H}$. Set $\hat{H} = \hat{P}\hat{X}$ and solve for $P = \hat{H}(\hat{X})^{-1}$. P is the linear-time-invariant equivalent of the specific $W \varepsilon \mathcal{W}$ picked, with respect to the specific trio of acceptable output vectors picked. Repeat over different trios. Repeat the entire operation over different $w \varepsilon \mathcal{W}$, giving a class $\mathcal{P}_1 = \{P\}$ which is the linear-time-invariant equivalent of the $\mathcal{W}$ family, with respect to the class of acceptable outputs $\mathcal{H}$ for input vector $\bar{r}_1$. Repeat the entire operation for $\bar{r}_2, \ldots, \bar{r}_{60}$ giving $\{\mathcal{P}_i\} = P_{total}$ which is the linear time equivalent for the nonlinear $\mathcal{W}$, with respect to the tribe of 60 families of acceptable output sets. The equivalence is exact if the conditions for application of Schauder's theorem are satisfied. We now have a linear time-invariant uncertain mio problem, which let us presume we can solve. If and only if we can guarantee the solution of the latter, then the same compensation functions will work for the original nonlinear uncertain mio plant. Hence the importance of quantitative linear time invariant design techniques (over and above their intrinsic importance)--for they enable the precise solution of nonlinear uncertainty problems.

The design effort in the above appears to be enormous but it is conceptually straightforward and easy. An ordinary control engineer can implement it and the digital computer is, of course, an essential

tool. Conceptually too, it appears possible to extend the method to obtain nonlinear relations between inputs and outputs within specified bounds, despite large plant uncertainty, even nonlinear time-varying, as can be done for the single input-output case. The prospect is fascinating. Imagine being able to work with the actual nonlinear equations of a jet engine, or a chemical process, etc., include uncertainties in the modelling, even uncertainty in system order (see Appendix), and designing to achieve outputs within specified tolerances over the given range of uncertainty.

6. DISTURBANCE ATTENUATION

Let x in Fig. 1 be a n x 1 matrix of distrubances. The resulting system output (with r = 0) is $c = (I + PG)^{-1} Px \triangleq Zx$, $Z = [z_{ig}]$, the n x n disturbance response matrix. Bounds on Z are given in the form

$$|z_{uv}(j\omega)| < b_{uv}(\omega) \quad , \forall \quad m \epsilon \mathcal{M} \tag{21}$$

Rewrite $c = Zx$ in the form $(P^{-1} + G)c = x$. Let $x_i \neq 0$ only for $i = v$, so $c_i = z_{iv}x_v$, and

$$\sum_{i=1}^{n} (p_{ui} + g_{ui})z_{iv} = \delta_v^u = \begin{pmatrix} 0, & u \neq v \\ 1, & u = v \end{pmatrix} \quad ,$$

$$(p_{uu} + g_{uu})z_{uv} = \delta_v^u - \sum_{i \neq u} (p_{ui} + g_{ui})z_{iv}$$

$$z_{uv} = \frac{\delta_v^u - \sum_{i \neq u} (p_{ui} + g_{ui})z_{iv}}{q} \tag{22}$$

Let

$$x_{uve}(\omega) \triangleq \sup_{m} \sum_{i \neq u} \left| \frac{P_{ui} + g_{ui}}{P_{uu}} \right| b_{iv}(\omega) \tag{23}$$

The $g_{ui}(\omega)$ $(i \neq u)$ can be chosen to minimize $x_{uve}(\omega)$, but for simplicity we shall assume them zero. From (22,23)

$$|z_{uv}(\omega)| < \left| \frac{\delta_{v}^{u}/P_{uu} + x_{uve}}{\left(1 + \frac{g_{uu}}{P_{uu}}\right)} \right| \tag{24}$$

If $1/P_{uu}$ satisfies the constraints listed, then it is obviously possible to guarantee $|z_{uv}(\omega)| <$ any finite number, no matter how small, at any finite ω. Also it can be made zero at a finite number of ω values by assigning poles to g_{uu} at these values. Assume that g_{uu} can be chosen to satisfy (21) $\forall$ $\omega \in [0,\infty)$. Then one can set up the conditions for Schauder's theorem, precisely as was done in 2.1. The set $b_{uv}(\omega)$ must have been formulated such that $B(n^2)$, the n^2 product set of the $b_{uv}(\omega)$, is compact convex in $C(n^2)$, analogous to $H(n^2)$ in 2.1. The analog of Φ in (7) must be formulated with the modification of Sec. 3.1, inasmuch as we do not care if the $z_{uv}(s)$ are nonminimum-phase.

Conditions analogous to (12-17) for $n = 2$, are obtained as follows. As $\omega \to \infty$, $g_{uu}/P_{uu} \to 0$ so in (24), the right side $\to$ its numerator. But $|z_{uv}(j\omega)| \leq b_{uv}(\omega)$ of (21). Let $u = 1$, $v = 2$ and then $u = 2$, $v = 1$ and obtain the necessary condition (for $g_{12} = g_{21} = 0$),

$$\text{As } \omega \to \infty, \quad p_{12}p_{21} < p_{11}p_{22}, \ \forall\ m \in \mathcal{M} \tag{25}$$

similar to (17) but here only at ∞, because there is no concern with the minimum-phase property. Setting $u = v = 1$, and then $u = v = 2$ in (24), we get the conditions

$$\text{As } \omega \to \infty,\ b_{11} > \left|\frac{1}{P_{11}}\right| = \left|p_{11} - \frac{p_{12}p_{21}}{p_{22}}\right|,\ b_{22} > \left|p_{22} - \frac{p_{12}p_{21}}{p_{11}}\right| \tag{26}$$

But in reality as $\omega \to \infty$, $c \to Px$ so $Z \to P$ and $z_{11} \to p_{11}$, $z_{22} \to p_{22}$. Hence, assignment of b_{ii} (as $\omega \to \infty$) to satisfy (25) is no obstacle, because the $b_{uv}(\omega)$ are upper bounds on the $|z_{uv}(j\omega)|$.

APPENDIX 1

EXISTENCE THEOREM FOR SINGLE-LOOP DESIGN

The plant transfer function p(s) is uncertain, belonging to a set $\mathcal{P} = \{p(s)\}$ and is imbedded in a two-degree-of-freedom single-loop feedback structure, as in Fig. 2 (p in place of p_{uve}). The rational functions f(s), g(s) (replacing f_{uv}, g_{uu} in Fig. 2) are to be chosen to satisfy specified tolerances on the command frequency-response $t(j\omega) = \frac{c(j\omega)}{r(j\omega)}$ and disturbance frequency response $t_d(j\omega) = c(j\omega)/d(j\omega)$, (r, d, c replacing r_v, $-d_{uv}$, c_u in Fig. 2).

A1. Constraints on $\mathcal{P}$

1. p(s) is a rational function with a fixed excess $e \geq 1$ of poles over zeros (this is relaxed later in A6, 7). $\exists$ at least one $p \varepsilon \mathcal{P}$ one of which is designated as p_o, all of whose poles and zeros are in the interior of the left half-plane.

2. At each $\omega \varepsilon [0,\infty)$, $\exists \inf_{\mathcal{P}} |p(j\omega)| \triangleq b(\omega) > 0$. $\exists \inf_{I} b(\omega) \triangleq b_I > 0$ for any finite interval $I = [0,\omega]$. Also, $|p_o|$ of A1(1) has a sup on each finite interval $I = [0,\omega]$, $\sup_I |p_o| = x_{oI}$.

3. As $s \to \infty$, $p(s) \to k_p/s^e$, $k_p \varepsilon [k_1, k_2]$ with $\infty > k_2 > k_1 > 0$, uniformly on $\mathcal{P}$ in the following sense: For any $\varepsilon > 0$, no matter how small, $\exists\ \omega_\varepsilon$ (independent of p(s)), such that for each $p \varepsilon \mathcal{P}$ there is associated a $k_p \varepsilon [k_1, k_2]$ so that

$$\left| \ln \left| \frac{p}{k_p/\omega^e} \right| \right| < \varepsilon \text{ and } \operatorname{Arg} \left| p(j\omega) + e\frac{\pi}{2} \right| < \varepsilon, \text{ for } \omega > \omega_\varepsilon .$$

Note that A1(1) permits changes in plant order, e.g., $\frac{1+\alpha T_1 s}{1+T_1 s}$ with say $\alpha \epsilon [2, 5]$, $T_1 \epsilon [0, 3]$. A1(2) dictates minimum-phase p(s) and that the $j\omega$ axis is not a limit of any sequence of p(s) zeros. A1(3) requires a uniform bound on the poles and zeros of all $p \epsilon P$.

A2. <u>Tolerances on $|t(j\omega)|$ and $|t_d(j\omega)|$</u>

(1) $0<A(\omega) \leqslant |t(j\omega)| \leqslant B(\omega)$ with $A, B \epsilon C\,[0, \infty)$, $\frac{B(\omega)}{A(\omega)} \geqslant \beta(\omega) > 1$. $\exists \inf_I \beta(\omega) = \beta_I > 1$ on any finite $I = [0,\omega]$.

(2) $\exists\ \lambda > 1.05$, ω_λ, $\ni$ for $\omega > \omega_\lambda$, $\frac{B(\omega)}{A(\omega)} = \frac{\lambda k_2}{k_1}$. This means that in the high ω range, the feedback is allowed to increase the sensitivity $S = \frac{\partial t(j\omega)/t(j\omega)}{\partial p(j\omega)/p(j\omega)}$, *rather than decrease it. In fact, as noted by* Bode, $\int_0^\infty \ln|S|d\omega = 0$ in any practical system, so the decrease in $S(|S|<1)$ achieved in the control bandwidth range, must be balanced by $|S|>1$ in another range. λ can be a large number, because as $\omega \to \infty$, $t(j\omega) \to 0$, e.g., suppose $k_2/k_1 = 10$, who cares if $|t(j\omega)| \epsilon [10^{-11}, 10^{-7}]$ $(\lambda = 10^3)$ at very large ω.

(3) The tolerances on $t_d(j\omega)$ are in the form $|t_d(j\omega)| \leqslant Q(\omega) > 0$. For any $I = [0, \omega]$, $\exists\ \inf Q(\omega) = \beta_{dI}$. Since $t_d = P(1 + pg)^{-1} = pS$ of A2(2), $|Q/p|>1$ at high frequencies, so $\exists\ \omega_d$, $\ni$ for $\omega > \omega_d$, $Q(\omega) = \beta_1(\omega)|p(j\omega|$, $\beta_1 > 1.05$.

stricted to the interior of the left half-plane $\forall p \varepsilon P$, and minimum-phase.

A3. <u>Choice of Compensation Functions</u>

Let $p_o(s)$ of A1(1) be the 'nominal' plant with $k_o \varepsilon [k_1, k_2]$ its associated k_p value of A1(3).

Let $\mathcal{E} = .01 \dfrac{k_o}{k_2}$ in A1(3), ω_t = largest of $\{\omega_\varepsilon, \omega_\lambda, \omega_d\}$, $I_t = [0, \omega_t]$,

$\gamma_t = \sup\limits_{I_t} \dfrac{|p_o(j\omega)|}{b_{I_t}}$, of A1(2). In Fig. 2,

$$t(s) = \frac{fgp}{1+gp} = \frac{f\ell_o}{\frac{p_o}{p} + \ell_o}, \quad \ell_o = g\, p_o. \tag{A1}$$

We want $\dfrac{\sup\limits_{\mathcal{P}} |t(j\omega)|}{\inf\limits_{\mathcal{P}} |t(j\omega)|} \leqslant \dfrac{B(\omega)}{A(\omega)}$ of A2(1). This is achieved in

$$I_t \text{ if } \frac{\sup\limits_{P} \left| \frac{p_o}{p} + \ell_o \right|}{\inf\limits_{P} \left| \frac{p_o}{p} + \ell_o \right|} < \beta_{I_t} \text{ of A2(1).}$$

Since $\left|\dfrac{p_o}{p}\right| < \gamma_t$ in I_t, it suffices for I_t, if $|\ell_o| > \gamma_t$ and $\dfrac{|\ell_o| + \gamma_t}{|\ell_o| - \gamma_t} \leqslant \beta_{I_t}$,

giving the sufficient condition

$$|\ell_o| \geqslant \gamma_t \frac{(\beta_{I_t} + 1)}{(\beta_{i_t} - 1)} \triangleq |\ell_{ot}|, \text{ in } I_t. \tag{A2}$$

To satisfy A2(3) in I_t, it is necessary that $|t_d| = \left|\frac{p}{1+gp}\right| =$

$\left|\frac{p_o}{\frac{p_o}{p} + \ell_o}\right| < Q(\omega)$, which is certainly achieved if

$$|\ell_o| > \frac{\sup|p_o|}{\beta_{dI_t}} + \frac{\sup|p_o|}{\inf|p|} = x_{oI_t}\left(\frac{1}{\beta_{dI_t}} + \frac{1}{b_{I_t}}\right) \triangleq |\ell_{od}|.$$

Therefore, choose

$$|\ell_o(j\omega)| > \text{larger of } (|\ell_{ot}|, |\ell_{od}|) \triangleq |\ell_{ox}|, \text{ in } I_t. \tag{A3}$$

Next, we find a bound on ℓ_o in $\bar{I}_t = [\omega_t, \infty)$ to satisfy A2 in $\bar{I}_t$. From A1(3), in $\bar{I}_t = [\omega_t, \infty)$, $\{\frac{-p_o}{p}\}$ lies in the narrow sliver V in Fig. A1, $\leq .01 \frac{k_o}{k_2} \leq .01$ radians angular width, with magnitude bounds $\left[.99 \frac{k_o}{k_2}, 1.01 \frac{k_o}{k_1}\right]$. Let A in Fig. A1 be a trial value of ℓ_o, so $\frac{p_o}{p} + \ell_o$ is the vector originating at point $\frac{p_o}{p}$ in V and terminating at A. Bounds on ℓ_o may be obtained so that

$$\frac{\sup_P \left|\frac{p_o}{p} + \ell_o\right|}{\inf_P \left|\frac{p_o}{p} + \ell_o\right|} \text{ satisfies A2(1,2) and A2(3) in } \bar{I}_t = [\omega_t, \infty).$$

It is easily seen that a very conservative boundary for ℓ_o in $\bar{I}_t$ is the vertical line $s = -\sigma$, with

$$\sigma = \sigma_1 = \frac{k_o}{2}\left(\frac{.99\lambda - 1.01}{\lambda k_2 - k_1}\right) > 0 \tag{A4a}$$

i.e., ℓ_o on the right of the line $s = -\sigma$, satisfies A2(1, 2) in $\bar{I}_t$.

$\left|\frac{p}{1+pq}\right| = \left|\frac{r_o}{\frac{p_o}{p} + \ell_o}\right| \leq \beta_1 |p|$, or $\left|\frac{r_o/p}{\frac{p_o}{p} + \ell_o}\right| \leq \beta_1$. This is easily satisfied if the above

$$\sigma = \sigma_2 = \frac{.99\,\frac{k_o}{k_2}\,(\beta_1 - 1)}{2\beta_1} \qquad \textbf{(A4b)}$$

Therefore, choose

$$\sigma = \text{smaller of } (\sigma_1, \sigma_2) \qquad \textbf{(A4c)}$$

Thus, the problem is to find $\ell_o(s)$ such that $|\ell_o(j\omega)|$ is outside the circle C in Fig. A1 for $\omega \leq \omega_t$ and to the right of the line $s = -\sigma$ for $\omega > \omega_t$. It is obviously very easy to find such an $\ell_o(s)$ which also has all its poles and zeros in the interior of the left half-plane, with any desired finite excess of poles over zeros, and which furthermore has the property shown in Fig. A1, i.e., lies on the right of $s \simeq -\sigma$, for all ω. For example, let

$$\ell_o = \frac{2\ell_{ox}}{\left(1 + \frac{s}{\omega_t}\right) \prod_1^e \left(1 + \frac{s}{\omega_i}\right)}$$

e any desired finite number, ω_1 = larger of $\left(10\omega_t, \frac{2\ell_{ox}\omega_t}{\sigma}\right)$, $\omega_{i+1} = 100\omega_i$.

Note that it would be impossible to guarantee the existence of the desired ℓ_o if p_o was nonminimum-phase (Horowitz and Sidi 1978).

It is conceivable that even though A2(1)-A2(3) are satisfied, A2(4) is not satisfied. Consider the zeros of $1 + pg = 1 + \ell_o \dfrac{p}{p_o}$ or of $(\frac{p_o}{p} + \ell_o)$. Recall that in $[0, \omega_t]$, $\{\frac{p_o}{p}\}$ lies in the circle of radius γ_t which is inside the circle C of Fig. A1, while ℓ_o lies outside the larger circle C. In $[\omega_t, \infty)$, ℓ_o lies on the right of the line $s = -\sigma$ while $\{\frac{-p_o}{p}\}$ is contained in V of Fig. A1. Also, $\ell_o(j\omega)$ lies on the right of $s = -\sigma$, $\forall\omega$. Hence, the vector $\frac{p_o}{p} + \ell_o$ does not encircle the origin clockwise (or alternatively ℓ_o does not so encircle $\frac{-p_o}{p}$), $\forall p \epsilon \mathcal{P}$, and the system is stable.

Application to mio system

In the mio system (4a), the loop function $\ell_{uu} = g_{uu}/P_{uu}$ must handle the n t_{uv} problems $v = 1,2,\ldots,n$. The bounds on ℓ_{uuo} will be, in general, different for each v with its own ℓ_{oxv}, I_{tv} of (A3) and σ_v of (A4c). Let $I_u = \max_v I_{tv}$, $\ell_{ou} = \max_v \ell_{oxv}$, $\sigma_u = \min_v \sigma_v$ be the design parameters for ℓ_{uuo}. Obviously such a ℓ_{uuo} is satisfactory for all n t_{uv} problems.

In Sec 3 (just before 3.1), there was noted the desirability of satisfying (6) on the boundary of a semicircle of radius ω_H in the right half-plane. This requires, in addition to the previous, rewriting Sec A3, replacing $j\omega$ by $\omega_H e^{j\theta}$, $\theta \epsilon [0, \pi/2]$. The development is easier if ω_H is large enough so that each $p_{ij} \doteq k_{ij}/s^{e_{ij}}$ on $s = \omega_H e^{j\theta}$. Clearly, there will emerge bounds on ℓ_{uuo} on $s = \omega_H e^{j\theta}$, which will have to be satisfied, in addition to those on $s = j\omega$. Obviously, such bounds can always be satisfied by suitable shaping of ℓ_{uuo}, so that $|\ell_{uuo}|$ is large enough on $\omega_H e^{j\theta}$.

There remains the design of $f(s)$, inasmuch as $\ell_o(s) = g(s)\, p_o(s)$ only determines $g(s)$. Note that $\ell_o(s)$ only guarantees that

$$\frac{\sup\limits_{P} \left|\frac{p_o}{p} + \ell_o\right|}{\inf\limits_{P} \left|\frac{p_o}{p} + \ell_o\right|} < \frac{B(\omega)}{A(\omega)} \text{ of A2(1);}$$

$f(s)$ is chosen so that $|t(j\omega)| \in [A(\omega), B(\omega)]$. For example, suppose $A(\omega_1) = .9$, $B(\omega_1) = 1.04$, and at ω_1, $\sup\limits_{\mathcal{M}} \left|\frac{p_o}{p} + \ell_o\right| = 100$, while $\inf\limits_{\mathcal{M}} \left|\frac{p_o}{p} + \ell_o\right| = 90$ with $|\ell_o(j\omega_1)| = 80$. The range of $|t(j\omega_1)| = \left|\frac{f\ell_o}{\frac{p_o}{p} + \ell_o}\right|$ is therefore $[.8|f(j\omega_1)|, .889|f(j\omega_1)|]$, so we need

$.8|f(j\omega_1)| > .9$, $.889|f(j\omega_1)| < 1.04$, giving the permissible range of $\left[\frac{9}{8}, \frac{1.04}{.889}\right]$ for $|f(j\omega_1)|$. In this way, the bounds on $|f(j\omega)|$ are found and it is always possible to find an $f(s)$ with left half-plane poles and zeros which satisfies such bounds.

The above procedure in all its details, is not recommended as a practical design procedure. Simplifications were made to make the proof easier, but the loop bandwidth is much larger than necessary. Its primary purpose is as an existence theorem. A practical optimum design procedure based on these ideas, but without the rigor, has been given in (Horowitz and Sidi 1972) and used a great deal with considerable success.

A5. Extensions

(1) It is possible to have $A(\omega) = B(\omega)$, $Q(\omega) = 0$ in A2(1), (3) at a finite number of ω values, by choosing g infinite at these points. The sensitivity zeros can be single or multiple.

(2) Some or all $p \varepsilon \mathcal{P}$ can have zerox on the $j\omega$ axis. If these zeros are precisely known (unlikely), g(s) can be assigned poles there. Otherwise, $t(j\omega)$ must be zero at these points for such p, requiring obviously much more careful statement of the tolerance on $t(j\omega)$ and $t_d(j\omega)$ near such points.

(3) The most significant extension is that Constraint A1(1) can be relaxed. There can be uncertainty in the order of the plant due to disappearing poles and zeros--closely related to the problem of singular perturbations (Porter and Tsingas 1978).

A6. Disappearing Poles and Zeros

Let

$$p = \frac{p_1 \prod^{m} (1+sa_i)}{\prod^{n} (1+sb_j)} = p_1 \Psi(s) \tag{A5}$$

with $a_i \varepsilon [0, a_{ix}]$, $b_j \varepsilon [0, b_{jx}]$ and $p_1 \varepsilon \mathcal{P}$ satisfying A1. The question of concern is: "For what m, n values can the loop transmission be arbitrarily large over an arbitrarily large bandwidth but still be practical, i.e., go to zero as $\omega \to \infty$ with any desired finite excess of poles over zeros?" For such m, n <u>any</u> tolerances satisfying A2 but otherwise arbitrary, can be satisfied.

In Fig. 2 and using (A5)

$$t(s) = \frac{fgp}{1 + gp} = \frac{fgp_1\Psi}{1 + gp_1\Psi} = \frac{f\ell}{1 + \ell} . \tag{A6}$$

The question posed can be answered by referring to the logarithmic complex plane (Nichols chart) in Fig. A2. The intersections of the zero db line with the vertical lines $(2n+1)\ 180^\circ$, $n = 0 \pm 1, \ldots$ is the point -1. Because of uncertainty, $\ell = gp_1\Psi$ is not a point but a set $\{\ell\}$ denoted here as the template of ℓ, $\mathcal{J}_\ell(\omega)$ which occupies some region in the complex plane--Fig. A2. The shape of $\mathcal{J}_\ell$ is that of $\{p_1\ \Psi\}$ because there is no uncertainty in g. The latter permits the translation (but not rotation of ℓ) in the complex logarithmic plane, horizontally by arg g and vertically by $|g|$ in db. For some finite ω range, large $|\ell|$ is needed so $\mathcal{J}_\ell$ lies above the zero db line, e.g., $\mathcal{J}_\ell(\omega_1)$ in Fig. A3. At large enough ω, $|\ell|$ must be very small ($\to 0$ as $\omega \to \infty$) so $\mathcal{J}_\ell$ must be well below the zero db lines and continue downwards to $-\infty$. In the transition of $\mathcal{J}_\ell$ from above to below the zero db line, it must not interest -1, nor encircle it. Hence, the width of $\mathcal{J}_\ell$ must be restricted to $<360^\circ$ for some ω interval in which $\mathcal{J}_\ell$ can squeeze in between two -1 points on its way downward (Fig. A2). But we want arbitrary sensitivity reduction for arbitrary bandwidth. This requires $\exists\ \omega_H, \ni$ for $\omega > \omega_H$ the width of $\mathcal{J}_\ell(\omega) < 360^\circ$.

Consider now the shape of $\mathcal{J}_\ell$ which is that of $\mathcal{J}_{p1\Psi}$ which is that of $\mathcal{J}_{p1} + \mathcal{J}_\Psi$ in the Nichols chart. Constraint A1(3) assures that at large enough ω the width of $\mathcal{J}_{p1} \to$ zero degrees, so it is entirely a question of the width of $\mathcal{J}_\Psi$. Consider any factor of Ψ, e.g., $\frac{1}{1 + j\omega b}$, $b \in [0, m]$. At any specific ω, the maximum width of the template is $\tan^{-1} \omega b_{max} = \tan^{-1} \omega m$. For $\omega m = 20$, the template is OAB in Fig. A3 (O at $b = 0$, B at $\omega b = 20$); for $\omega m = 100$ it is OABU and for $\omega m = 1000$, it is OABUV. For two such independent factors, it is easy to find the new template. This is done in Fig. A3 for $\omega m_1 = \omega m_2 = 20$. Draw O"A"B" = OAB and position O" at points on OAB (because of the independent uncertainties). The result is ABEJDEFO $= \mathcal{J}[(\frac{1}{1+j\omega b_1})(\frac{1}{1+j\omega b_2})]$, b_1, $b_2 \in [0, m]$. The template of a zero factor $(1+j\omega a)$ is obtained by reflecting that of the pole if $(\omega a)_{max} = (\omega b)_{max}$, giving OA'B' in Fig. 3.

As ω increases, the contribution of each factor (pole or zero) $\to 90^o$ in width. Therefore, while theoretically four factors can be admitted between two -1 points, stability margins dictate a maximum of three. But this is only a necessary condition, because it must also be possible to decrease $|L|$ from arbitrary large to small values. This basically means that over arbitrary large frequency range Arg L must be overwhelmingly negative. The extreme right side of $\mathcal{J}_\ell$ must then lie on the left of the 0^o line in Fig. A2. One disappearing pole or zero poses no problem, but two do because the left side of $\mathcal{J}_\ell$ will then intersect -1

be made rigorous, of course.) Note that this is for the most demanding situation. If only stability is desired, then any finite number of disappearing poles or zeros can be handled.

APPENDIX 2

A7. CONDITIONS THAT Φ MAPS $H(n^2)$ INTO ITSELF

The constraints on H_{uv} were given in 2.1. The first is satisfied if (6) is. From Appendix 1, it is seen that at finite ω, the only possible difficulty is, if in Fig. 2, $d_{uv}(j\omega)$ is unbounded but $g_{uu}p_{uve}$ is bounded. If such unboundedness of d_{uv} is at a finite number of ω values, it is possible to assign poles there to g_{uu}. Since d_{uv} is a function of the plant parameters (Eq. 4b), an infinite number of such poles is conceivable, in fact is so in practice if there is one, because of inevitable uncertainties. How can d_{uv} be unboundeo while p_{uve} is bounded? From (3,5)

$$d_{uv} = \frac{\sum\limits_{i \neq u} (-1)^{iu} \Delta_{iu} t_{iv}}{\Delta = \sum\limits_{i \neq u} (-1)^{iu} p_{iu}\Delta_{iu} + p_{uu}\Delta_{uu}}, \quad p_{uve} = \frac{\Delta}{\Delta_{uu}} \tag{A7}$$

Hence, such a situation is possible, if Δ_{iu} has such a pole which is cancelled by a zero of p_{iu} and is not present in $p_{uu}\Delta_{uu}$ e.g. $n = 2$, p_{21} has a pole at $\pm ja$, p_{12} has a zero there and $p_{11}p_{22}$ does not have a pole there. Such situations are therefore not allowed. It can be argued that they are in practice impossible (Horowitz and Shaked 1975),

because p_{12}, p_{21} must involve different physical parameters and the uncertainties in each cannot be 100% correlated. Irrespective of this argument, the constraint is

$$|P_{ij}(j\omega)| \text{ is uniformly bounded on } [0,\infty) \text{ and } M, \ \forall\, i \neq j \tag{A8}$$

The problem is more difficult as $\omega \to \infty$ and is treated in Section 3.

To satisfy constraint (2) on H_{uv} in 2.1, note first that if $z(j\omega) = z_1 + jz_2$ with z_1, z_2 real functions of ω, then $|z|'$ is bounded if z' is bounded. For

$$|z|' = (z_1^2 + z_2^1)' = \frac{z_1 z_1' + z_2 z_2^1}{z_1^2 + z_2^2} \quad . \quad \text{If } z^1 = z_1^1 + jz_2^1$$

is bounded, so are z_1^1, z_2^1 and $|z|'$. Consider separately $\tau_{uv}'(j\omega)$ and $(\tau_{duv}\, d_{uv})'(j\omega)$ of (4). The former can be written

$$\tau' = \left(\frac{f\ell}{1+\ell}\right)' = f' \frac{\ell}{1+\ell} + \frac{f(gp' + g'p)}{(1+gp)^2} \tag{A9}$$

$$\ell = gp, \quad p = p_{uve}, \quad g = g_{uu}$$

of Fig. 2. Since $f\ell/(1+\ell)$ is bounded by the appropriate B_{uv}, the first term on the right of (A9) needs only uniform boundedness of f', which is easy as f is chosen by the designer. Obviously, the only possible difficulties with the second term of (A9) are g', p' which can be infinite only at $j\omega$ axis poles. However, at such poles, the denominator forces the second term to be bounded. At large ω where gp, g', $p' \to 0$, there is obviously no problem.

Next, consider $(\tau_{duv} d_{uv})'(\omega)$ written as

$$y' = \left(\frac{dp}{1+gp}\right)' = \frac{d'p}{1+gp} + \frac{d(p' - g'p^2)}{(1+gp)^2} \tag{A10}$$

g, p of (A9), $d = d_{uv}$.

From (A8), d and since d is a rational function d', are uniformly bounded on the $j\omega$ axis. From the previous discussion re p', g' etc. and Appendix 1, y' can be uniformly bounded on any finite ω range by proper choice of g. At large ω each of d, d', p, p' etc $\to 0$.

It is easy to see that constraint (3) is satisfied, because it has been required that the elements of P, G, F all $\to 0$ as $s \to \infty$.

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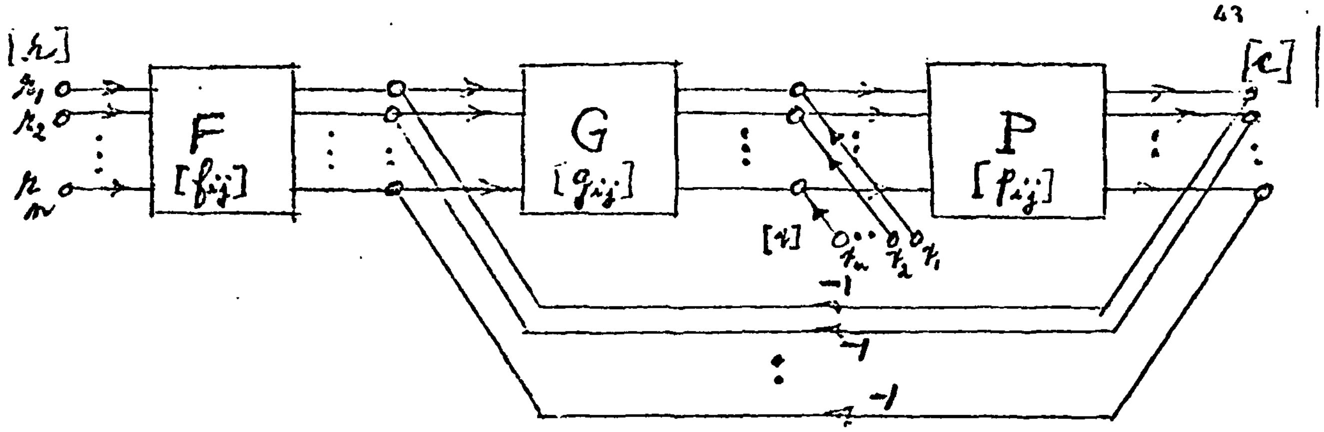


Fig. 1 Multiple input-output two matrix degree-of-freedom feedback structure $c = Tr$, $T = [t_{ij}]$, $c = [c_1 \ldots c_n]'$, $r = [r_1 \ldots r_n]'$.

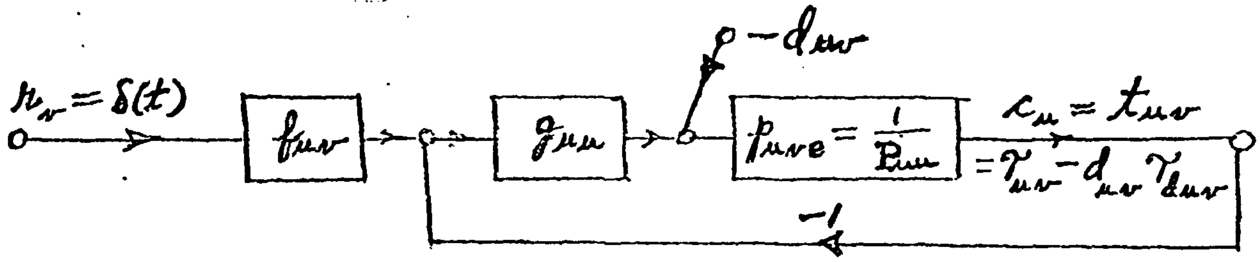


Fig. 2 Single-loop structure equivalent, for synthesis of t_{uv};

$d_{uv} = \sum_{i\neq u} P_{ui} t_{iv}$, $P^{-1} = [P_{ij}]$, $P = [p_{ij}]$

$$\tau_{uv} = \frac{f_{uv} g_{uu} p_{uve}}{1 + g_{uu} p_{uve}}, \quad \tau_{duv} = \frac{p_{uve}}{1 + g_{uu} p_{uve}}.$$

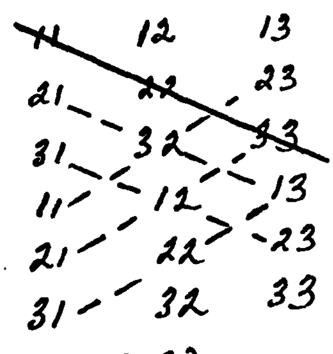


FIG. 3a

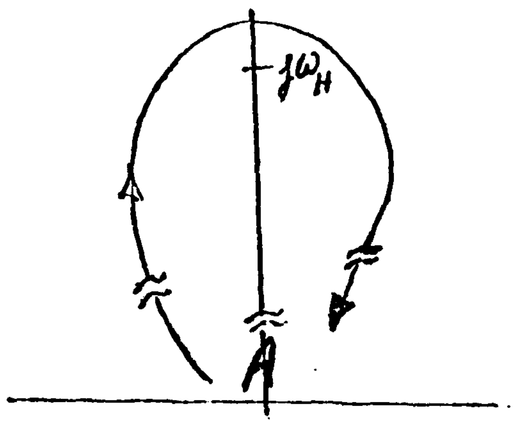
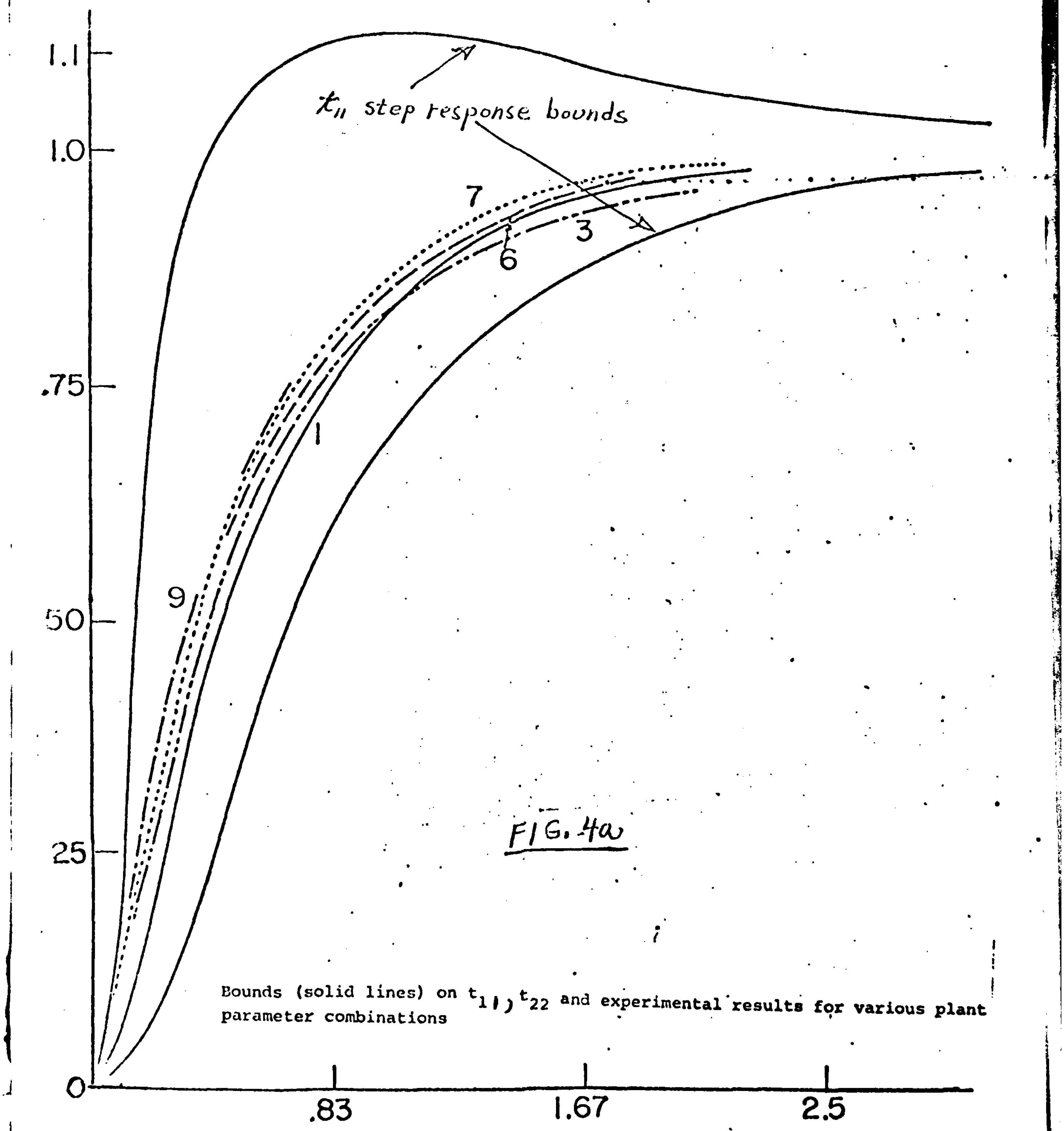


Fig. 3b To reach A in right half-plane, a zero must cross $j\omega$ axis above $j\omega_H$.



Bounds (solid lines) on t_{11}, t_{22} and experimental results for various plant parameter combinations

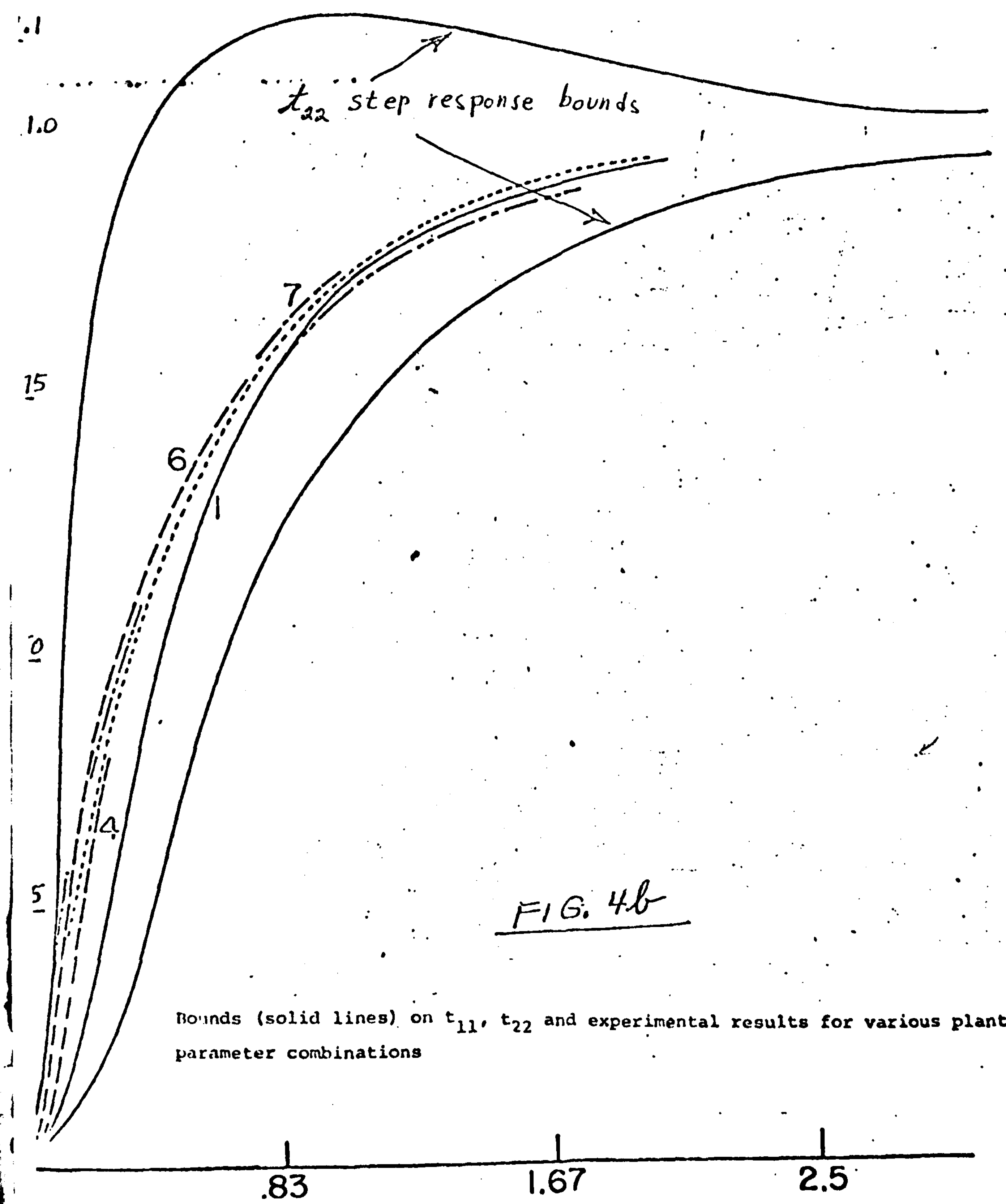


FIG. 4b

Bounds (solid lines) on t_{11}, t_{22} and experimental results for various plant parameter combinations

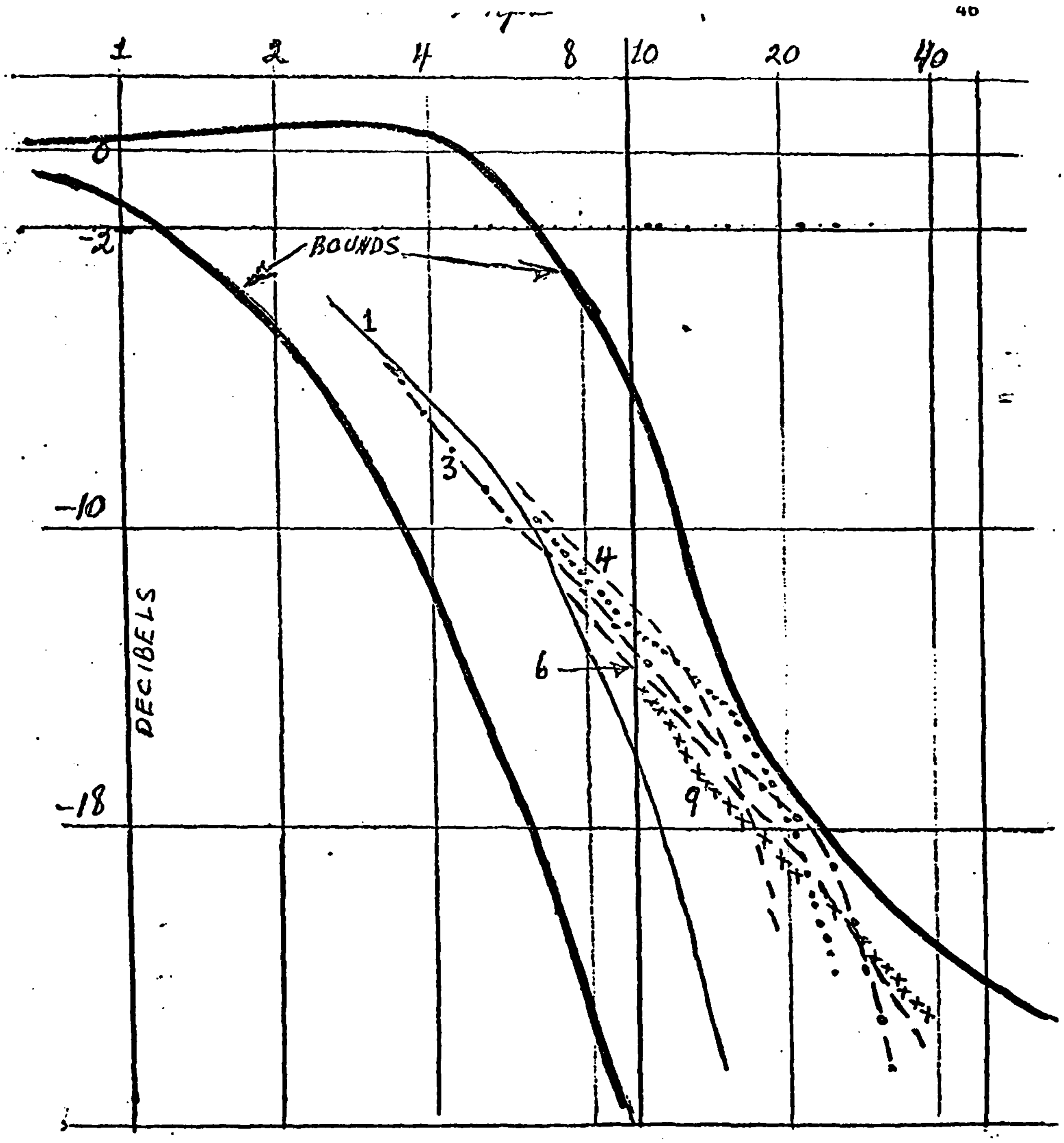


Fig. 5. "Equivalent" frequency domain bounds and experimental results for various plant parameter sets

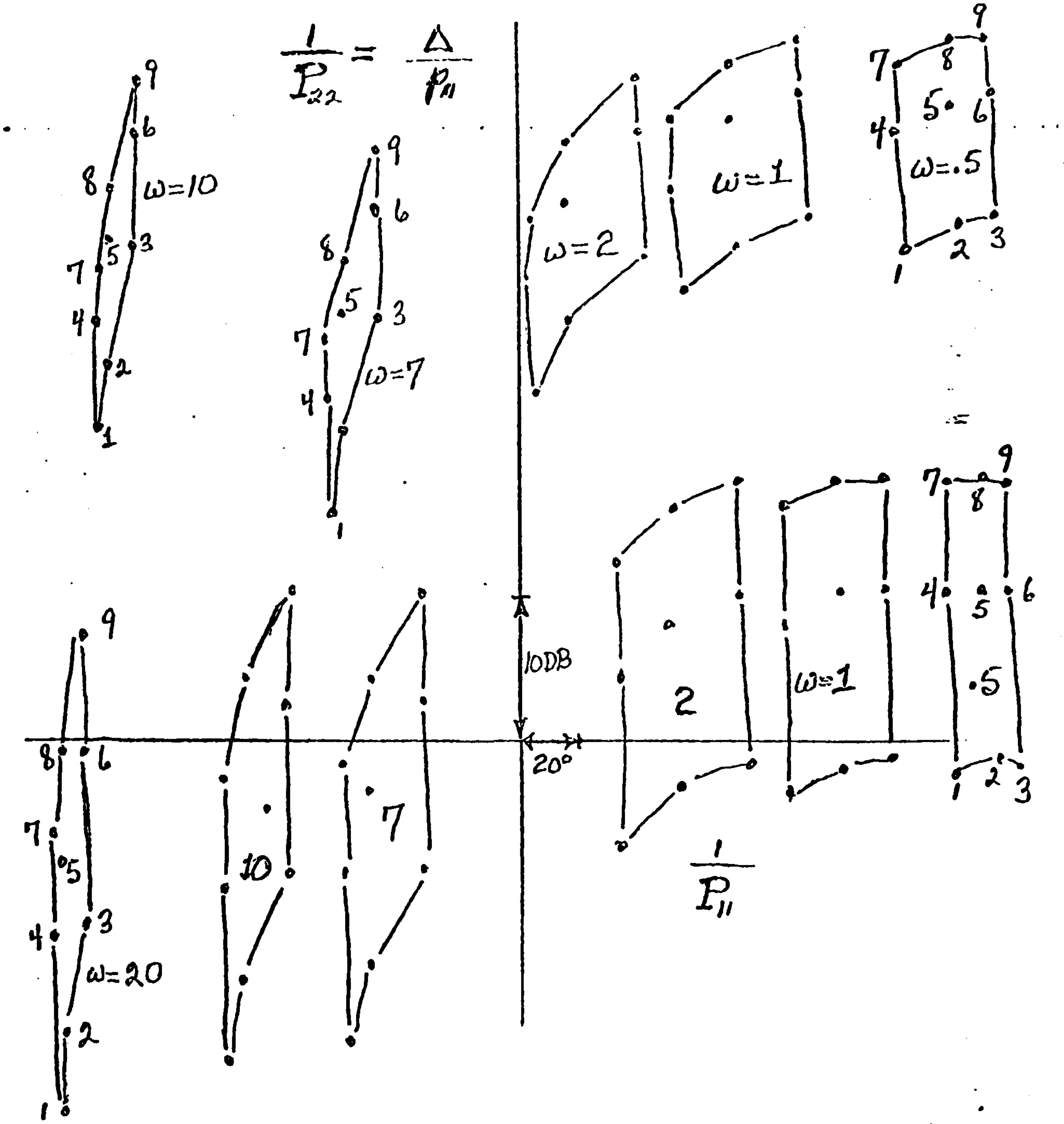


Fig. 6. Templates of $P_{22}^{-1} = \Delta/p_{11}$, $P_{11}^{-1} = \Delta/p_{22}$ at various frequencies, on Nichols Chart

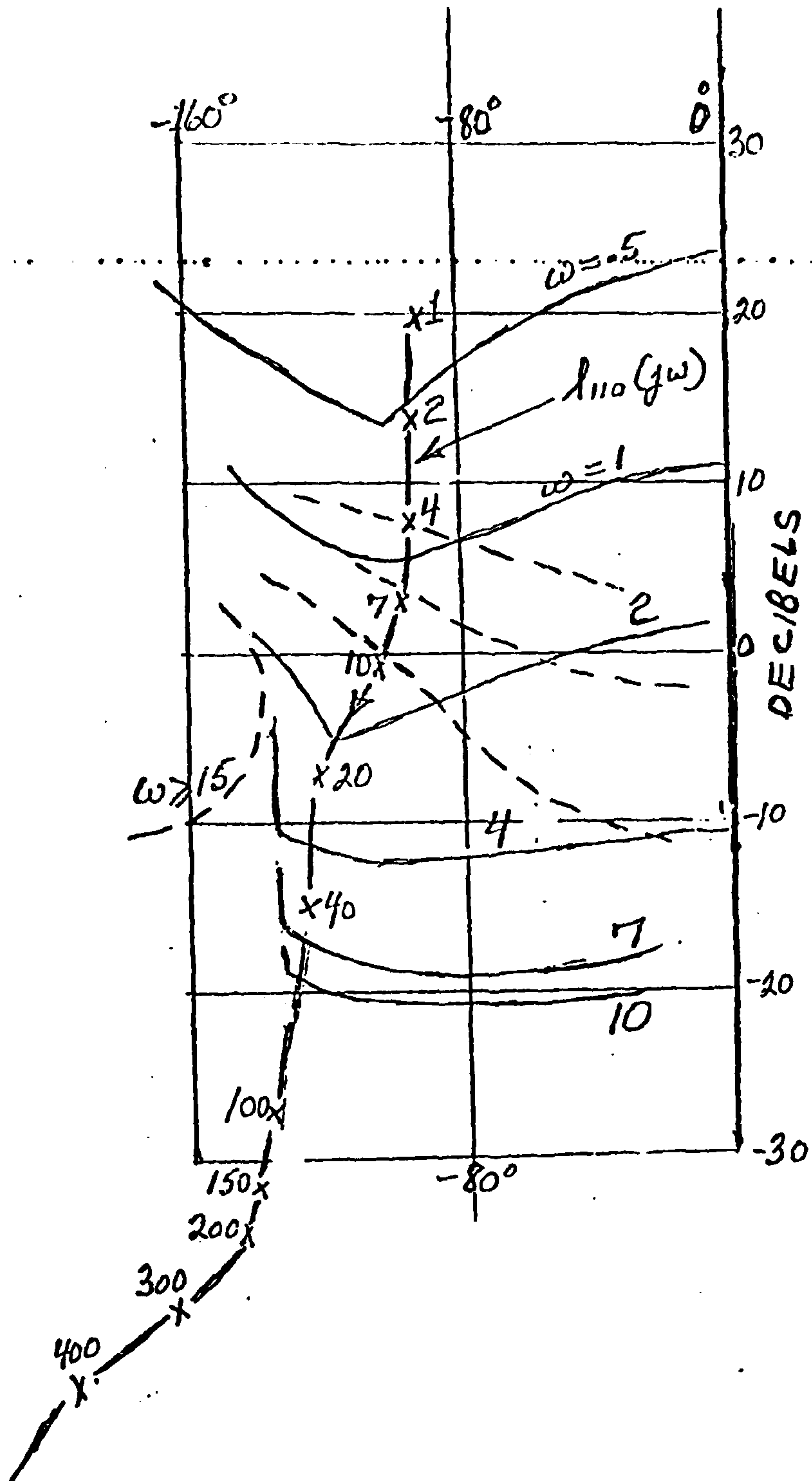


Fig. 7. Bounds on $\ell_{110}(j\omega)$. Solid lines are due to $|\tilde{\gamma}_{11}|$, broken lines due to $|\gamma_{12}\, d_{12e}|$

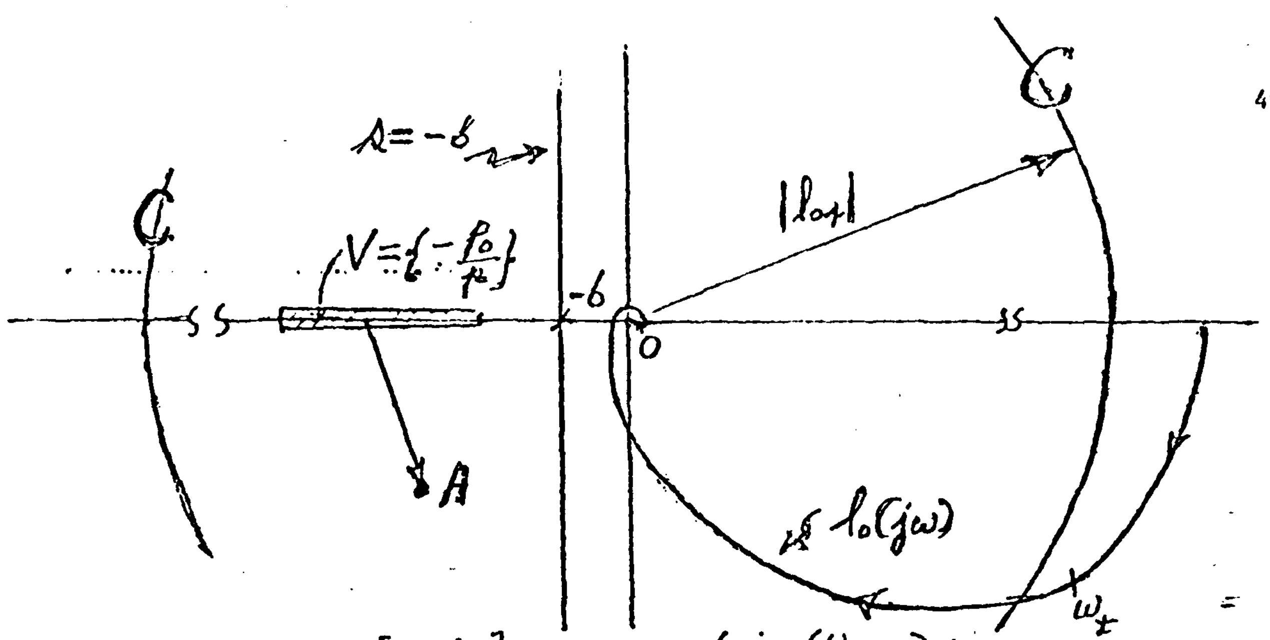


Fig. A1. Bounds on ℓ_0: C in $[0, \omega_t]$, $\ell = -b$ in (ω_t, ∞); and a satisfactory $\ell_0(j\omega)$.

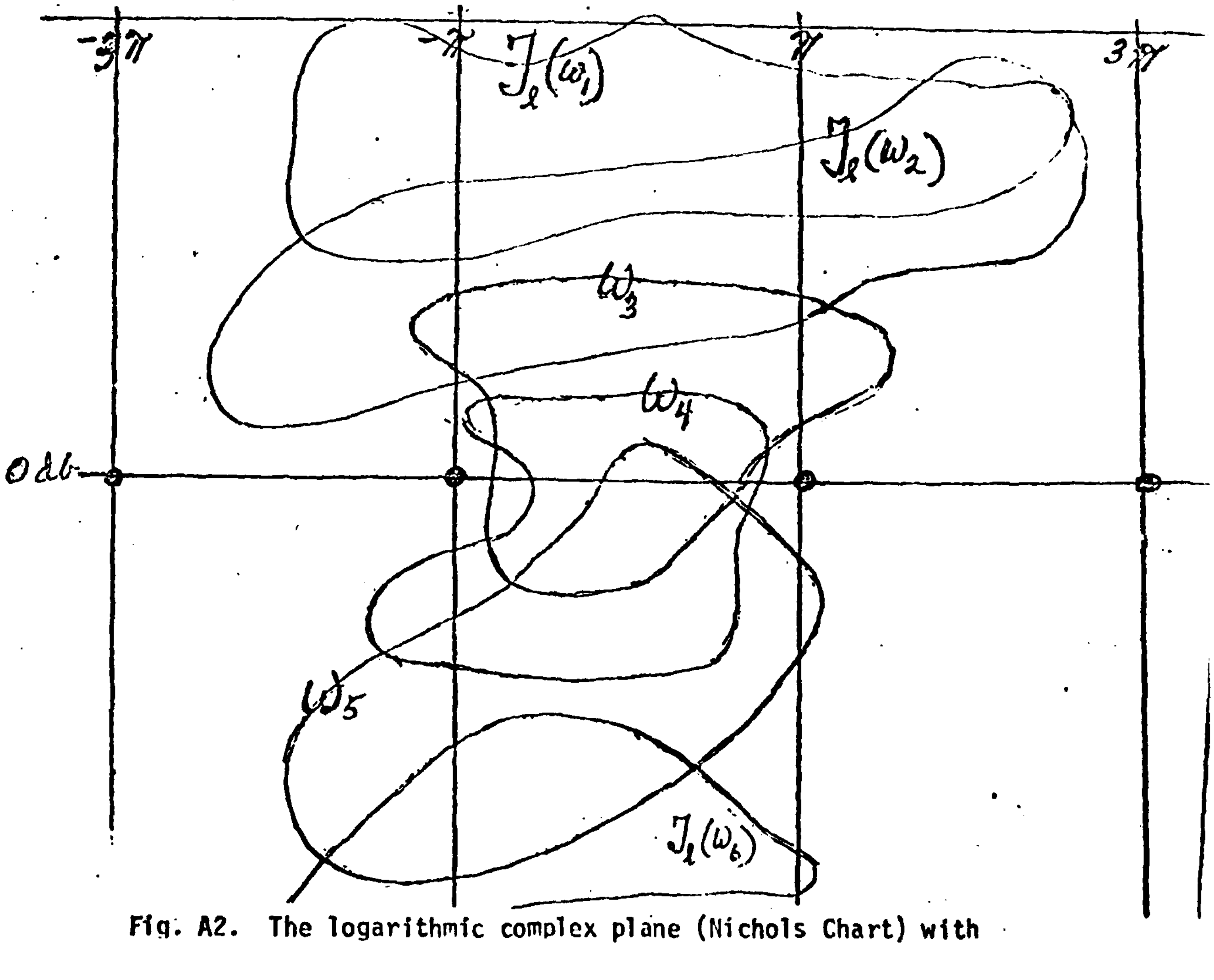


Fig. A2. The logarithmic complex plane (Nichols Chart) with

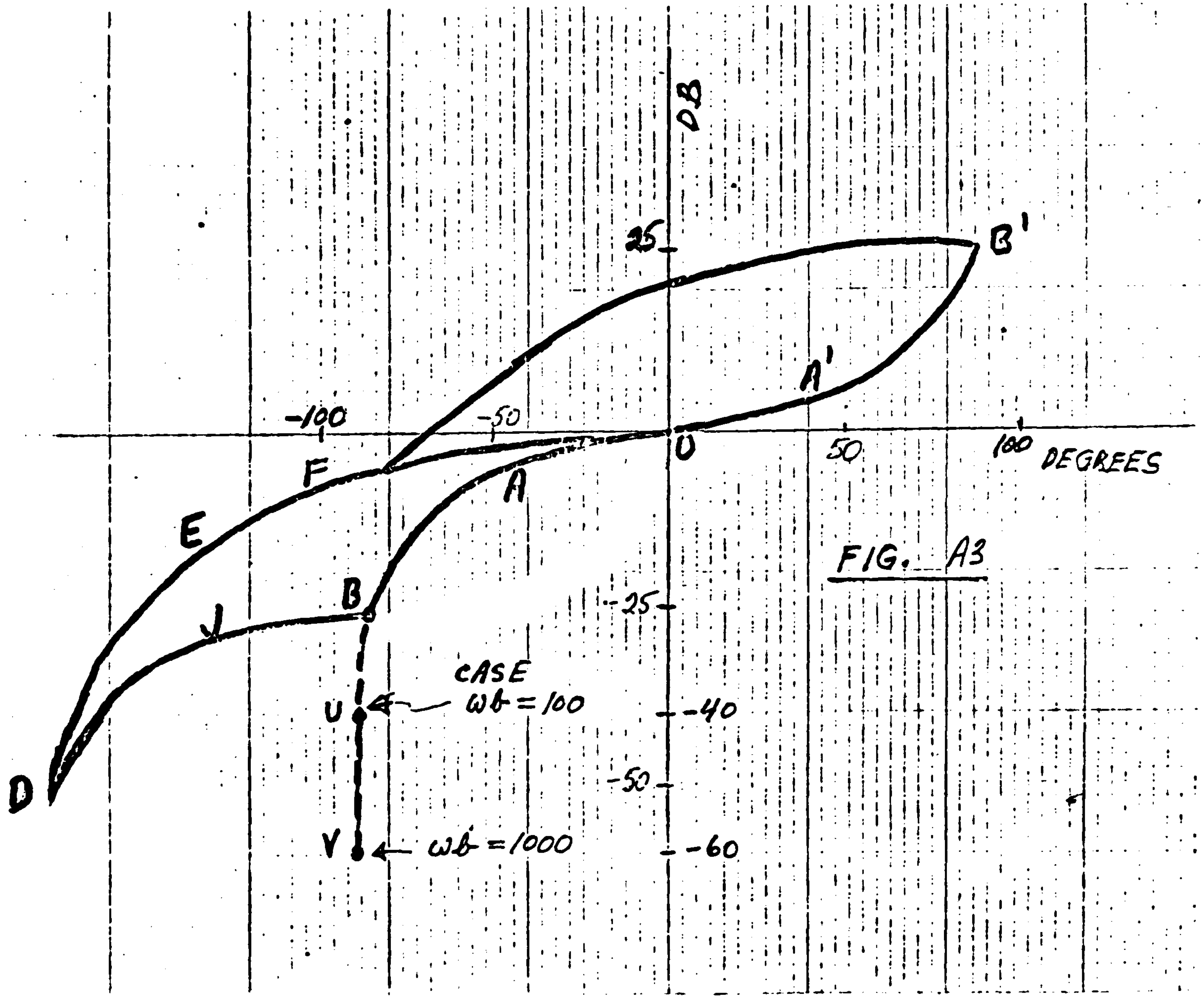

DB
25
B'
A'
-100
-50
0
50
100
DEGREES
F
A
E
B
J
-25
CASE
ωb = 100
U
-40
D
-50
V
ωb = 1000
-60

FIG. A3

FILM

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